Strategic Trading with Higher Order Beliefs

Jungsuk Han, Pete Kyle and Mina Lee

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Question

- The standard REE model has a strong assumption that agents share homogeneous beliefs. What happens if we relax this assumption?

Roadmap

- Model setup and solution method.
- Equilibrium solution with general belief structures.
  - Belief inconsistencies are more important than disagreements.
- Example: heterogeneous beliefs on the noise trading.
  - Common prior with infinite orders of beliefs: standard REE.
  - When agents consider up to finite orders of beliefs only, the price can become more informative.
  - Exogenous noise trading can be interpreted as higher order uncertainty in the liquidity demand quantity.
- Conclusion
- Remaining questions

Setup (1/3)

- Kyle (1989):
  - A one-period model with N privately informed strategic traders and noise traders.
  - Strategic: taking into account their price impact.
    - ★ In contrast to the competitive model à la Hellwig (1980) ("schizophrenia" problem).
  - Traders submit their demand schedules (a set of demand quantities conditional on the market clearing price, i.e., the limit order book).
- Our setup:
  - Allows informed traders to have heterogeneous beliefs about the model primitives and heterogeneous beliefs about other traders' beliefs about the model primitives, etc.
  - Focuses on heterogeneous expectations rather than precisions.

Setup (2/3)

- Time: $t \in \{0, 1\}$
- Assets:
  - One risky asset with liquidation value $\tilde{v}$.
  - One storage technology.
- Agents:
  - N traders
    - CARA preferences with absolute risk aversion A.
    - Endowed with private information:

$$\tilde{i}_n = \tilde{v} + \tilde{e}_n, \ \forall n. \tag{1}$$

  - Noise traders submit exogenous demand $\tilde{z}$.
- All random variables are normally distributed; and $\tilde{v}$, $\tilde{e}_n$'s and $\tilde{z}$ are independent with each other.

Setup (3/3)

- Traders may have heterogeneous expectations on
  - Prior distribution of the liquidation value
$$\tilde{v} \sim N\left(E^n\{\tilde{v}\}, \tau_0^{-1}\right). \tag{2}$$
  - Bias on his signal
$$\tilde{e}_m \sim N\left(E^n\{\tilde{e}_m\}, \tau_e^{-1}\right). \tag{3}$$
  - Noise trading
$$\tilde{z} \sim N\left(E^n\{\tilde{z}\}, \tau_z^{-1}\right). \tag{4}$$
  - Other agents' expectations, their expectations of others' expectations, etc.
$$E^m\{E^n\{\cdot\}\} \neq E^n\{\cdot\}, \text{ for some } m \neq n. \tag{5}$$
- All heterogeneous expectations are independent with $\tilde{v}$.

Solution Technique

- Two stages:
  1. Solve for an equilibrium while taking an arbitrary set of beliefs as given.
  2. Substitute a specific set of beliefs into the equilibrium equations.
- Note that
  - Heterogeneous expectations themselves have no information.
  - Without the information asymmetry from private information, there is no role for higher order beliefs.
  - Higher order expectations play a role because they are used to infer the private information of other traders.
  - Trick: all variances are symmetric, and the heterogeneous expectations are summarized in the constant term.

Linear Price Conjecture

- Conjecture that the price satisfies the following equation:

$$P = \tilde{p}_0 + p_s \Sigma_{i=1}^{N} \tilde{i}_n + p_z \tilde{z}. \quad (6)$$

  - The constant $\tilde{p}_0$ is a random variable that incorporates higher order expectations.
  - With homogeneous precisions, agents have common knowledge on p_s and p_z.
  - Traders attempt to infer others' private information that is (partially) reflected on the equilibrium price.
  - The randomness in $\tilde{p}_0$ can make the inference problem more or less difficult.

- Agent n's demand depends on
  - Their private information i_n.
  - The information they learn from the market clearing price P.

Bayesian Updating

- The additional information agent n learns from the price is equivalent to

$$y_n \equiv \frac{P}{(N-1)\,p_s} - \frac{i_n}{N-1}$$

$$= \frac{\tilde{p}_0}{(N-1)\,p_s} + \frac{\sum_{m\neq n}\tilde{i}_m}{N-1} + \frac{p_z}{(N-1)\,p_s}\tilde{z}. \qquad (7)$$

- Therefore

$$E^n\{\tilde{v} \mid i_n, P\} = E^n\{\tilde{v} \mid i_n, y_n\}$$

$$= E^n\{\tilde{v}\} + \frac{\tau_e}{\tau_I}\cdot\left(\tilde{i}_n - E^n\{\tilde{i}_n\}\right) + \frac{(N-1)\,\varphi_I\tau_e}{\tau_I}\cdot\left(y_n - E^n\{\tilde{y}_n\}\right), \qquad (8)$$

where the price informativeness $\varphi_I \in [0,1]$ is defined as

$$\left[Var^n\{\tilde{v} \mid i_n, P\}\right]^{-1} = \tau_I = \tau_0 + \tau_e + \varphi_I\,(N-1)\,\tau_e. \qquad (9)$$

Optimal Demand Schedule

- Agent n's optimal demand schedule is

$$X_n(i_n, P) = \frac{E^n\{\tilde{v} \mid i_n, y_n\} - P}{A\tau_I^{-1} + \alpha\lambda}. \quad (10)$$

  - To allow the comparison with the competitive case, we use $\alpha\lambda$ instead of λ. (λ is the actual price impact; while $\alpha\lambda$ is the perceived price impact by all agents.)
  - When $\alpha = 0$, the model corresponds to a (schizophrenic) competitive equilibrium.
  - When $\alpha = 1$, agents correctly take into account their price impact.
  - $E^n\{\tilde{v} \mid i_n, y_n\}$ reflects (1) Agent n's own expectation on the fundamental, the bias on his signal, the biases on others' signals and the noise trading, and (2) Agent n's expectation on the constant $\tilde{p}_0$.

Market Clearing Condition

- Now we clear the market:

$$\Sigma_{n=1}^{N} X_n(i_n, P) + \tilde{z} = 0. \tag{11}$$

- We get the price as a linear function of
  - The coefficients of the initial price conjecture p_s and p_z.
  - All agents' first-order expectations on $\tilde{p}_0$ and the model primitives.
  - Other equilibrium variables and parameters.
- Then in an equilibrium,
  - The initial linear price conjecture is correct.
  - The price informativeness is given by

$$\frac{1}{(N-1)\varphi_I \tau_e} = Var^n\{\tilde{y}_n\} = Var^n\left\{\frac{\Sigma_{m\neq n}\tilde{i}_m}{N-1} + \frac{\tilde{p}_0}{(N-1)p_s} + \frac{p_z\tilde{z}}{(N-1)p_s}\right\}. \tag{12}$$

Equilibrium $\tilde{p}_0$ (1/2)

- First define $\bar{E}\{\cdot\}$ as the average expectation.

$$\bar{E}\{\cdot\} = \frac{1}{N}\Sigma_{n=1}^{N} E^n\{\cdot\}. \tag{13}$$

  - $\bar{E}\{\cdot\}$ is a linear operator.
  - When $N > 1$, $\bar{E}\{\cdot\}$ may not satisfy the law of iterated expectations.

- The equilibrium constant p_0 satisfies

$$p_0 = \frac{\tau_0}{\tau_I}\bar{E}\{\tilde{v}\} - \frac{\tau_e}{\tau_I}\frac{\Sigma_{n=1}^{N} E^n\{\tilde{e}_n\}}{N} - \frac{\varphi_I(N-1)\tau_e}{\tau_I}\frac{\Sigma_{n=1}^{N} E^n\{\Sigma_{m\neq n}\tilde{e}_m\}}{N(N-1)}$$
$$-\left[\left(\frac{\alpha}{\frac{N-2}{N-1} - 2\varphi_I}\right) + \left(\frac{1-\alpha}{1-\varphi_I}\right)\right]\varphi_I A \tau_I^{-1}\bar{E}\{\tilde{z}\}$$
$$+\left(\frac{N\varphi_I}{1+(N-1)\varphi_I}\right)(p_0 - \bar{E}\{p_0\}). \tag{14}$$

Equilibrium $\tilde{p}_0$ (2/2)

- Even if agents form their demand using only first order expectations, the equilibrium price reflects infinite orders of heterogeneous expectations.
- $\tilde{p}_0$ is determined as

$$\tilde{p}_0 = \Sigma_{k=0}^{\infty} \left(\frac{N\varphi_I}{1+(N-1)\varphi_I} \right)^k \left(I - \bar{E}\right)^k \times$$

$$\left(\frac{\tau_0}{\tau_I} \bar{E}\left\{\tilde{v}\right\} - \frac{\tau_e}{\tau_I} \frac{\Sigma_{n=1}^{N} E^n \left\{\tilde{e}_n\right\}}{N} - \frac{\varphi_I (N-1) \tau_e}{\tau_I} \frac{\Sigma_{n=1}^{N} E^n \left\{\Sigma_{m\neq n} \tilde{e}_m\right\}}{N(N-1)} \right)$$

$$- \frac{\varphi_I A \tau_I^{-1}}{\left[\left(\frac{\alpha}{\frac{N-2}{N-1} - 2\varphi_I} \right) + \left(\frac{1-\alpha}{1-\varphi_I} \right) \right]^{-1}} \cdot \Sigma_{k=0}^{\infty} \left(\frac{N\varphi_I}{1+(N-1)\varphi_I} \right)^k \left(I - \bar{E}\right)^k \bar{E}\left\{\tilde{z}\right\}. \tag{15}$$

Disagreements vs. Inconsistencies

- Note that the price reflects the average expectation and the "inconsistencies" in average expectations. We define the average inconsistency of the first order as

$$\left(I-\bar{E}\right)\bar{E}\left\{\cdot\right\}=\bar{E}\left\{\cdot\right\}-\bar{E}\left\{\bar{E}\left\{\cdot\right\}\right\} \tag{16}$$

  and the average inconsistency of K-th order as

$$\left(I-\bar{E}\right)^{K}\bar{E}\left\{\cdot\right\}=\left(I-\bar{E}\right)^{K-1}\bar{E}\left\{\cdot\right\}-\bar{E}\left\{\left(I-\bar{E}\right)^{K-1}\bar{E}\left\{\cdot\right\}\right\}. \tag{17}$$

- Then

$$\Sigma_{k=0}^{\infty}\left(I-\bar{E}\right)^{k}\bar{E}\left\{\cdot\right\}=\bar{E}\left\{\cdot\right\}+\Sigma_{k=1}^{\infty}\left(I-\bar{E}\right)^{K}\bar{E}\left\{\cdot\right\}. \tag{18}$$

- With agreeing to disagree, this series collapses to the average expectation.

$$\Sigma_{k=0}^{\infty}\left(I-\bar{E}\right)^{k}\bar{E}\left\{\cdot\right\}=\bar{E}\left\{\cdot\right\}. \tag{19}$$

- What matters for the equilibrium price is not the disagreement itself but the inconsistencies.

Example: Common Prior

- Now focus on heterogeneous beliefs on the noise trading. Suppose that before the trading game begins, all agents
  - Have common prior on noise trading $\tilde{z} \sim N\left(0, \tau_{0Z}^{-1}\right)$.
  - Receive private signals

$$\tilde{s}_n \sim \tilde{z} + \tilde{\varepsilon}_n, \text{ where } \tilde{\varepsilon}_n \sim N\left(0, \tau_S^{-1}\right) \text{ and } \tau_Z = \tau_{0Z} + \tau_S. \tag{20}$$

- Then

$$E^n\{\tilde{z}\} = \frac{\tau_S}{\tau_Z} s_n, \text{ and } \bar{E}\{\tilde{z}\} = \frac{\tau_S}{\tau_Z}\left(\frac{\sum_{n=1}^N s_n}{N}\right), \tag{21}$$

$$E^n\{E^m\{\tilde{z}\}\} = E^n\left\{\frac{\tau_S}{\tau_Z} s_m\right\} = \left(\frac{\tau_S}{\tau_Z}\right)^2 s_n, \ \forall m \neq n, \tag{22}$$

and

$$\left(I - \bar{E}\right)^k \bar{E}\{\tilde{z}\} = \left(\frac{N-1}{N}\right)^k \left(1 - \frac{\tau_S}{\tau_Z}\right)^k \left(\frac{\tau_S}{\tau_Z}\right) \frac{\sum_{n=1}^N s_n}{N}. \tag{23}$$

Finite Orders of Beliefs

- Substituting the heterogeneous expectations into the general solutions is equivalent to solving for an equilibrium with the standard technique with two kinds of information.
- The standard REE solution does not require agents to have the same expectations and incorporates infinite orders of expectations.
  - The key is common prior and infinite orders of inconsistencies.
- What if agents do not think about all infinite orders but instead only up to Kth order?

$$\frac{4(N-1)\tau_e}{A^2}(1-\varphi_I)\left(\frac{N-2}{2(N-1)}-\varphi_I\right)^2 =$$

$$\frac{\tau_Z+(N-1)\varphi_I^2\tau_s}{[\tau_Z+(N-1)\varphi_I\tau_s]^2}\varphi_I \times \left[1-\left(\frac{(N-1)\varphi_I}{1+(N-1)\varphi_I}\right)^K\left(1-\frac{\tau_S}{\tau_Z}\right)^K\right] \quad (24)$$

Noise Trading as Higher Order Uncertainty

- Instead of assuming exogenous noise trading, suppose there is a deterministic quantity z of liquidity trading.
- However, agents have dogmatic (heterogeneous) beliefs about the quantity z, and are unsure about what other agents think.
- The uncertainties in higher order beliefs work in a similar fashion to the exogenous trading.
  - This creates liquidity and makes agents to trade.
  - Without noise trading, there is an equilibrium that is not fully revealing.
  - How? Choose parameters so that two equilibria share the price informativeness φ_I:

$$\sigma_Z^2 = \frac{\tau_Z + (N-1)\,\varphi_I^2 \tau_s}{[\tau_Z + (N-1)\,\varphi_I \tau_s]^2} \times \left[1 - \left(\frac{(N-1)\,\varphi_I}{1+(N-1)\,\varphi_I} \right)^K \left(1 - \frac{\tau_S}{\tau_Z} \right)^K \right] \tag{25}$$

- Noise trading may be inside everybody's head!

Conclusion

- Summary
  1. Trading model with information asymmetry can be extended to allow heterogeneous (higher order) expectations.
  2. The inconsistencies in beliefs can be more important than the disagreements in determining the equilibrium price.
  3. A standard REE concept incorporates infinite orders of beliefs. Forcing agents to consider finite orders only can make the price more informative.
  4. An exogenous noise trading can be interpreted as agents' higher order uncertainty in the liquidity demand.
- To do list
  - What are general properties of higher order beliefs that makes an equilibrium well-defined?
  - How do we think about higher order beliefs outside the common prior setup? What does being rational mean in a practical sense?

Convergence

- What are general properties of higher order beliefs that makes an equilibrium well-defined?

$$\Sigma_{k=0}^{\infty}\left(\frac{N\varphi_I}{1+(N-1)\varphi_I}\right)^k\left(I-\bar{E}\right)^k\bar{E}\{\cdot\} \tag{26}$$

or

$$\Sigma_{k=0}^{\infty}\left(I-\bar{E}\right)^k\bar{E}\{\cdot\} \tag{27}$$

- For $N=1$,

$$\Sigma_{k=0}^{\infty}\left(I-\bar{E}\right)^k\bar{E}\{\cdot\}=\bar{E}\{\cdot\}. \tag{28}$$

- With common prior,

$$\left(I-\bar{E}\right)^k\bar{E}\{\cdot\}=ar^k,\ \text{where } |r|<1. \tag{29}$$

- When $\lim_{k\to\infty}\left(I-\bar{E}\right)^k=0$,

$$\Sigma_{k=0}^{\infty}\left(I-\bar{E}\right)^k\bar{E}\{\cdot\}=\bar{E}^{-1}\bar{E}\{\cdot\}=I? \tag{30}$$