

Increases in Risk Aversion and the Distribution of Portfolio Payoffs*

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Abstract

In this paper, we derive new comparative statics results in the distribution of portfolio payoffs using stochastic dominance. Specifically, an agent is less risk averse than another if and only if the agent chooses a payoff that is distributed as the other's payoff plus a nonnegative random variable plus conditional-mean-zero noise for all state-price-density distributions. Additionally, if either agent has non-increasing absolute risk aversion, the non-negative random variable can be chosen to be constant. This main result also holds in some special incomplete markets with two assets or two-fund separation. In multiple periods, increasing risk aversion has an ambiguous impact at a point in time, but there is a natural mixture of distributions over time that preserves our results.

JEL Classification Codes: D33, G11.

Keywords: risk aversion, portfolio theory, stochastic dominance, complete markets, two-fund separation

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I. Introduction

The trade-off between risk and return arises in many portfolio problems in finance. This trade-off is more-or-less assumed in mean-variance optimization, and is also present in the comparative statics for two-asset portfolio problems explored by Arrow (1965) and Pratt (1964) (for a model with a riskless asset) and Kihlstrom, Romer, and Williams (1981) and Ross (1981) (for a model without a riskless asset). However, the trade-off is less clear in portfolio problems with many risky assets, as pointed out by Hart (1975). Assuming a complete market with many states (and therefore many assets), we show that a less risk-averse (in the sense of Arrow and Pratt) agent's portfolio payoff is distributed as the payoff for the more risk-averse agent, plus a non-negative random variable (extra return), plus conditional-mean-zero noise (risk). Therefore, the general complete-markets portfolio problem, which may not be a mean-variance problem, still trades off risk and return.

If either agent has non-increasing absolute risk aversion, then the non-negative random variable (extra return) can be chosen to be a constant. We also give a counter-example that shows that in general, the non-negative random variable cannot be chosen to be a constant. In this case, the less risk averse agent's payoff can also have a higher mean and a lower variance than the more risk averse agent's payoff. We further prove a converse theorem. Suppose there are two agents, such that in all complete markets, the first agent chooses a payoff that is distributed as the second's payoff, plus a non-negative random variable, plus conditional-mean-zero noise. Then the first agent is less risk averse than the other agent.

Our main result can be extended to a multiple period model. Consumption at each date may not be ordered when risk aversion changes, due to shifts in the timing of consumption. However, for agents with the same pure rate of time preference, there

is a weighting of probabilities across periods that preserves the single-period result.

Our main result also extends to some special settings with incomplete markets, for example, a two-asset world with a risk-free asset. The proof is in two parts. The first part is the standard result: decreasing the risk aversion increases the portfolio allocation to the asset with higher return. The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. However, for a two-asset world without a risk-free asset, both parts of the proof fail in general and we have a counter-example. Therefore, our result is not true in general with incomplete markets. We further provide sufficient conditions under which our results still hold in a two-risky-asset world using Ross's stronger measure of risk aversion. Each result from two assets can be re-interpreted as applying to parallel settings with two-fund separation identifying the two funds with the two assets.

The proofs in the paper make extensive use of results from stochastic dominance, portfolio choice, and Arrow-Pratt and Ross (1981) risk aversion. One contribution of the paper is to show how these concepts relate to each other. We use general versions of the stochastic dominance results for L^1 random variables¹ and monotone concave preferences, following Strassen (1965) and Ross (1971). To see why our results are related to stochastic dominance, note that if the first agent's payoff equals the second agent's payoff plus a non-negative random variable plus conditional-mean-zero noise, this is equivalent to saying that negative the first agent's payoff is monotone-concave dominated by negative the second agent's payoff.

Section II introduces the model setup and provides some preliminary results, Sec-

¹We assume that the consumptions have unbounded distributions instead of compact support (e.g., Rothschild-Stiglitz (1970)). Compact support for consumption is not a happy assumption in finance because it is violated by most of our leading models.

tion III derives the main results. Section IV extends the main results in a multiple-period model. Section V discusses the case with incomplete markets. Section VI illustrates the main results using some examples and Section VII concludes.

II. Model Setup and Some Standard Results

We want to work in a fairly general setting with complete markets and strictly concave increasing von Neumann-Morgenstern preferences. There are two agents A and B with von Neumann-Morgenstern utility functions $U_A(c)$ and $U_B(c)$, respectively. We assume that $U_A(c)$ and $U_B(c)$ are of class C^2 , $U'_A(c) > 0$, $U'_B(c) > 0$, $U''_A(c) < 0$ and $U''_B(c) < 0$. Each agent's problem has the form:

Problem 1 Choose random consumption \tilde{c} to

$$\begin{aligned} \max E[U_i(\tilde{c})], \\ \text{s.t. } E[\tilde{p}\tilde{c}] = w_0. \end{aligned} \tag{1}$$

In Problem 1, $i = A$ or B indexes the agent, w_0 is initial wealth (which is the same for both agents), and $\tilde{p} > 0$ is the state price density. We will assume that \tilde{p} is in the class \mathcal{P} for which both agents have optimal random consumptions with finite means, denoted \tilde{c}_A and \tilde{c}_B .

The first order condition is

$$U'_i(\tilde{c}_i) = \lambda_i \tilde{p}, \tag{2}$$

i.e., the marginal utility is proportional to the state price density \tilde{p} . We have

$$\tilde{c}_i = I_i(\lambda_i \tilde{p}), \tag{3}$$

where I_i is the inverse function of $U_i'(\cdot)$. By continuity and negativity of the second order derivative $U_i''(\cdot)$, \tilde{c}_i is a decreasing function of $\tilde{\rho}$.

Our main result will be that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where “ \sim ” denotes “is distributed as,” $\tilde{z} \geq 0$, and $E[\tilde{\varepsilon}|c_B+z] = 0$.² We firstly review and give the proofs in the Appendix of some standard results in the form needed for the proofs of our main results.

Lemma 1 *If B is weakly more risk averse than A , $\left(\forall c, -\frac{U_B''(c)}{U_B'(c)} \geq -\frac{U_A''(c)}{U_A'(c)}\right)$, then*

1. *for any solution to (2) (which may not satisfy the budget constraint (1)), there exists some critical consumption level c^* (can be $\pm\infty$) such that $\tilde{c}_A \geq \tilde{c}_B$ when $\tilde{c}_B \geq c^*$, and such that $\tilde{c}_A \leq \tilde{c}_B$ when $\tilde{c}_B \leq c^*$;*
2. *assuming \tilde{c}_A and \tilde{c}_B have finite means, and A and B have equal initial wealths, then $E[\tilde{c}_A] \geq E[\tilde{c}_B]$.*

The first result in Lemma 1 implies that the consumptions function of the more risk averse agent crosses that of the less risk averse agent at most once and from above. This single-crossing result can be derived from Pratt (1964) and it can also be found in Gollier (2001). Lemma 1 gives us a sense in which decreasing the agent’s risk aversion takes us further from the riskless asset. In fact, we can obtain a more explicit description (our main result) of how decreasing the agent’s risk aversion changes the optimal portfolio choice. The description and proof are both related to monotone concave stochastic dominance.³ The following theorem gives a distributional char-

²Throughout this paper, the letters with “tilde” denote random variables, and the corresponding letters without “tilde” denote particular values of these variables.

³We avoid using the concept of second order stochastic dominance in this paper because it is known that there are two different definitions. In this paper, we follow unambiguous terminology from Ross (1971): (1) if $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all nondecreasing functions, then \tilde{X} monotone stochastically dominates \tilde{Y} ; (2) if $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave functions, then \tilde{X} concave stochastically dominates \tilde{Y} ; (3) if $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave nondecreasing functions, then \tilde{X} monotone-concave stochastically dominates \tilde{Y} .

acterization of stochastic dominance for all monotone and concave functions of one random variable over another. The form of this result is from Ross (1971) and is a special case of a result of Strassen (1965) which generalizes a traditional result for bounded random variables to possibly unbounded random variables with finite means.

Theorem 1 (*Monotone Concave Stochastic Dominance: Strassen (1965) and Ross (1971)*) Let \tilde{X} and \tilde{Y} be two random variables defined in R^1 with finite means; then $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$, for all concave nondecreasing functions $V(\cdot)$, i.e., \tilde{X} monotone-concave stochastically dominates \tilde{Y} , if and only if $\tilde{Y} \sim \tilde{X} - \tilde{Z} + \tilde{\varepsilon}$, where $\tilde{Z} \geq 0$, and $E[\tilde{\varepsilon}|X - Z] = 0$.

Rothschild and Stiglitz (1970, 1972) popularized a similar characterization of stochastic dominance for all concave functions (which implies equal means) that is a special case of another result of Strassen's.

Theorem 2 (*Concave Stochastic Dominance: Strassen (1965), and Rothschild and Stiglitz (1970, 1972)*) Let \tilde{X} and \tilde{Y} be two random variables defined in R^1 with finite means; then $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$, for all concave functions $V(\cdot)$, i.e., \tilde{X} concave stochastically dominates \tilde{Y} , if and only if $\tilde{Y} \sim \tilde{X} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|X] = 0$.

III. Main Results

Suppose agent A with utility function U_A and agent B with utility function U_B have identical initial wealth w_0 and solve Problem 1. We have

Theorem 3 *If B is weakly more risk averse than A in the sense of Arrow and Pratt ($\forall c, -\frac{U_B''(c)}{U_B'(c)} \geq -\frac{U_A''(c)}{U_A'(c)}$), then for every $\tilde{\rho} \in \mathcal{P}$, \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where*

$\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. Furthermore, if $\tilde{c}_A \neq \tilde{c}_B$, neither \tilde{z} nor $\tilde{\varepsilon}$ is identically zero.

Proof: The first step of the proof⁴ is to show that $-\tilde{c}_B$ monotone-concave stochastically dominates $-\tilde{c}_A$, i.e., $E[V(-\tilde{c}_B)] \geq E[V(-\tilde{c}_A)]$ for any concave nondecreasing function $V(\cdot)$. By Lemma 1, \tilde{c}_A and \tilde{c}_B are monotonely related and there is a critical value c^* above which \tilde{c}_A is weakly larger and below which \tilde{c}_B is weakly larger. Let $V'(\cdot)$ be any selection from the subgradient correspondence $\nabla V(\cdot)$, then $V'(\cdot)$ is positive and nonincreasing and it is the derivative of $V(\cdot)$ whenever it exists. Recall that the subgradient for concave⁵ $V(\cdot)$ is $\nabla V(x_1) \equiv \{s | (\forall x), V(x) \leq V(x_1) + s(x - x_1)\}$ in the definition of Rockafellar (1970). By concavity of $V(\cdot)$, $\nabla V(x)$ is nonempty for all x_1 . And if $x_2 > x_1$, then $s_2 \leq s_1$ for all $s_2 \in \nabla V(x_2)$ and $s_1 \in \nabla V(x_1)$.

The definition of subgradient for concave $V(\cdot)$ implies that

$$V(x + \Delta x) \leq V(x) + V'(x)\Delta x. \quad (4)$$

Letting $x = -\tilde{c}_B$ and $\Delta x = -\tilde{c}_A + \tilde{c}_B$ in (4), we have

$$V(-\tilde{c}_A) - V(-\tilde{c}_B) \leq V'(-\tilde{c}_B)(-\tilde{c}_A + \tilde{c}_B). \quad (5)$$

If $\tilde{c}_B \geq c^*$, then $\tilde{c}_A \geq \tilde{c}_B$ (by Lemma 1), and $V'(-\tilde{c}_B) \geq V'(-c^*)$, while if $\tilde{c}_B \leq c^*$, then $\tilde{c}_A \leq \tilde{c}_B$ and $V'(-\tilde{c}_B) \leq V'(-c^*)$. In both cases, we always have $(V'(-\tilde{c}_B) -$

⁴Because the integral condition does not hold under unbounded distributions, a proof using Lemma 1 and the intergral condition would be wrong. More specifically, we cannot get that $-\tilde{c}_B$ monotone concave stochastically dominates $-\tilde{c}_A$ directly from $\int_{q=-\infty}^c [F_{-\tilde{c}_A}(q) - F_{-\tilde{c}_B}(q)] dq \geq 0$ for unbounded distributions.

⁵For convex $V(\cdot)$, the inequality is reversed.

$V'(-c^*)(\tilde{c}_A - \tilde{c}_B) \geq 0$. Rewriting (5) and substituting in this inequality, we have

$$V(-\tilde{c}_B) - V(-\tilde{c}_A) \geq V'(-\tilde{c}_B)(\tilde{c}_A - \tilde{c}_B) \geq V'(-c^*)(\tilde{c}_A - \tilde{c}_B). \quad (6)$$

Since $V(\cdot)$ is nondecreasing and $E[\tilde{c}_A] \geq E[\tilde{c}_B]$ (result 2 of Lemma 1), we have

$$E[V(-\tilde{c}_B) - V(-\tilde{c}_A)] \geq E[V'(-c^*)(\tilde{c}_A - \tilde{c}_B)] = V'(-c^*)(E[\tilde{c}_A] - E[\tilde{c}_B]) \geq 0. \quad (7)$$

Therefore, we have that $-\tilde{c}_B$ is preferred to $-\tilde{c}_A$ by all concave nondecreasing $V(\cdot)$, and by Theorem 1, this says that $-\tilde{c}_A$ is distributed as $-\tilde{c}_B - \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon} - c_B - z] = 0$. This is exactly the same as saying that \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z} + (-\tilde{\varepsilon})$, where $\tilde{z} \geq 0$ and $E[-\tilde{\varepsilon}|c_B + z] = 0$. Relabel $-\tilde{\varepsilon}$ as $\tilde{\varepsilon}$, and we have proven the first sentence of the theorem.

To prove the second sentence of the theorem, note that because \tilde{c}_A and \tilde{c}_B are monotonely related, \tilde{c}_A is distributed the same as \tilde{c}_B only if $\tilde{c}_A = \tilde{c}_B$. Therefore, if $\tilde{c}_A \neq \tilde{c}_B$, one or the other of \tilde{z} or $\tilde{\varepsilon}$ is not identically zero. Now, if \tilde{z} is identically zero, then $\tilde{\varepsilon}$ must not be identically zero, and \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{\varepsilon}$, by Jensen's inequality, we have $E[U_A(\tilde{c}_A)] = E[U_A(\tilde{c}_B + \tilde{\varepsilon})] = E[E[U_A(\tilde{c}_B + \tilde{\varepsilon})|\tilde{c}_B]] < E[U_A(E[\tilde{c}_B|\tilde{c}_B] + E[\tilde{\varepsilon}|\tilde{c}_B])] = E[U_A(\tilde{c}_B)]$, which contradicts the optimality of \tilde{c}_A for agent A . If $\tilde{\varepsilon}$ is identically zero, then \tilde{z} must not be, and \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z}$, where $\tilde{z} \geq 0$ and is not identically zero. Therefore, \tilde{c}_A strictly monotone stochastically dominates \tilde{c}_B , contradicting optimality of \tilde{c}_B for agent B . This completes the proof that if \tilde{c}_A and \tilde{c}_B do not have the same distribution, then neither $\tilde{\varepsilon}$ nor \tilde{z} is identically zero. *Q.E.D.*

We now prove a converse result of Theorem 3: if in all complete markets, one agent chooses a portfolio whose payoff is distributed as a second agent's payoff plus

a nonnegative random variable plus conditional-mean-zero noise, then the first agent is less risk averse than the second. Specifically, we have

Theorem 4 *If for all $\tilde{\rho} \in \mathcal{P}$, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$, then B is weakly more risk averse than A ($\forall c, -\frac{U_B''(c)}{U_B'(c)} \geq -\frac{U_A''(c)}{U_A'(c)}$). This implies a converse result of Theorem 3: if for all $\tilde{\rho} \in \mathcal{P}$, \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$, then B is weakly more risk averse than A .*

We prove this theorem by contradiction. If B is not weakly more risk averse than A , then there exists a constant \hat{c} , such that $-\frac{U_B''(\hat{c})}{U_B'(\hat{c})} < -\frac{U_A''(\hat{c})}{U_A'(\hat{c})}$. Since U_A and U_B are of the class of C^2 , from the continuity of $-\frac{U_i''(c)}{U_i'(c)}$, where $i = A, B$, we get that there exists an interval RA containing \hat{c} , s.t., $\forall c \in RA$, $-\frac{U_B''(c)}{U_B'(c)} < -\frac{U_A''(c)}{U_A'(c)}$. We pick $c_1, c_2 \in RA$ with $c_1 < c_2$. Now from Lemma 5 in the Appendix, there exists hypothetical agents A_1 and B_1 , so that U_{A_1} agrees with U_A and U_{B_1} agrees with U_B on $[c_1, c_2]$, but A_1 is everywhere strictly more risk averse than B_1 (and not just on $[c_1, c_2]$).

Fix any $\lambda_B > 0$ and choose $\tilde{\rho}$ to be any random variable that takes on all the values on $[\frac{U_B'(c_2)}{\lambda_B}, \frac{U_B'(c_1)}{\lambda_B}]$. Then, the corresponding \tilde{c}_B solving the first order condition $U_B'(\tilde{c}_B) = \lambda_B \tilde{\rho}$ takes on all the values on $[c_1, c_2]$. Because $U_B'' < 0$, the *F.O.C* solution is also sufficient (expected utility exists because $\tilde{\rho}$ and $U_B(\tilde{c}_B)$ are bounded), \tilde{c}_B solves the portfolio problem for utility function U_B , state price density $\tilde{\rho}$ and initial wealth $w_0 = E[\tilde{\rho}\tilde{c}_B]$. Since $U_{B_1} = U_B$ on the support of \tilde{c}_B , letting $\tilde{c}_{B_1} = \tilde{c}_B$, then \tilde{c}_{B_1} solves the corresponding optimization for U_{B_1} for $\lambda_{B_1} = \lambda_B$.

We now show that there exists λ_{A_1} such that $\tilde{c}_{A_1} \equiv I_{A_1}(\lambda_{A_1}\tilde{\rho})$ satisfies the budget constraint $E[\tilde{\rho}\tilde{c}_{A_1}] = w_0$. Due to the choice of U_{A_1} , $I_{A_1}(\lambda_{A_1}\tilde{\rho})$ exists and is a bounded random variable for all λ_{A_1} . Letting $\underline{\rho} = \frac{U_B'(c_2)}{\lambda_B}$ and $\bar{\rho} = \frac{U_B'(c_1)}{\lambda_B}$ (so, $\tilde{\rho} \in [\underline{\rho}, \bar{\rho}]$), we

define $\lambda_1 = \frac{U'_{A_1}(c_1)}{\underline{\rho}}$ and $\lambda_2 = \frac{U'_{A_1}(c_2)}{\bar{\rho}}$, then we have

$$c_1 = I_{A_1}(\lambda_1 \underline{\rho}) > I_{A_1}(\lambda_1 \tilde{\rho}), \quad c_2 = I_{A_1}(\lambda_2 \bar{\rho}) < I_{A_1}(\lambda_2 \tilde{\rho}). \quad (8)$$

The inequalities follow from $I_{A_1}(\cdot)$ decreasing. From (8) and $c_1 \leq \tilde{c}_B \leq c_2$, we have

$$E[\tilde{\rho} I_{A_1}(\lambda_1 \tilde{\rho})] < E[\tilde{\rho} c_1] \leq E[\tilde{\rho} \tilde{c}_B] = w_0, \quad E[\tilde{\rho} I_{A_1}(\lambda_2 \tilde{\rho})] > E[\tilde{\rho} c_2] \geq E[\tilde{\rho} \tilde{c}_B] = w_0. \quad (9)$$

Since $I_{A_1}(\lambda \tilde{\rho})$ is continuous from the assumption that $U_{A_1}(\cdot)$ is in the class of C^2 and $U''_{A_1} < 0$. By the intermediate value theorem, there exists λ_{A_1} , such that $E[\tilde{\rho} I_{A_1}(\lambda_{A_1} \tilde{\rho})] = w_0$, *i.e.*, \tilde{c}_{A_1} satisfies the budget constraint for $\tilde{\rho}$ and w_0 .

From the second result of Lemma 6 in the Appendix, if $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, then we have that \tilde{c}_{B_1} has a wider range of support than that of \tilde{c}_{A_1} . Let the support of A_1 's optimal consumption be $[c_3, c_4] \subseteq [c_1, c_2]$. From the construction of U_{A_1} , $U_{A_1} = U_A$ on the support of \tilde{c}_{A_1} . Letting $\tilde{c}_A = \tilde{c}_{A_1}$, then \tilde{c}_A solves the corresponding optimization for U_A for $\lambda_A = \lambda_{A_1}$.

Now, since B_1 is strictly less risk averse than A_1 , from Theorem 3, $\tilde{c}_{B_1} \sim \tilde{c}_{A_1} + \tilde{z}_1 + \tilde{\varepsilon}_1$, where $\tilde{z}_1 \geq 0$ and $E[\tilde{\varepsilon}_1 | c_{A_1} + z_1] = 0$. Furthermore, if $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, then neither \tilde{z}_1 nor $\tilde{\varepsilon}_1$ is identically zero. From the first result of Lemma 6 in the Appendix, if A_1 is strictly more risk averse than B_1 , then $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$. Thus, by Theorem 3, neither \tilde{z}_1 nor $\tilde{\varepsilon}_1$ is identically zero. Therefore, $E[\tilde{c}_{B_1}] > E[\tilde{c}_{A_1}]$, *i.e.* $E[\tilde{c}_B] > E[\tilde{c}_A]$, this contradicts the assumption that, for all $\tilde{\rho} \in \mathcal{P}$, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$. This also contradicts a stronger condition: for all $\tilde{\rho} \in \mathcal{P}$, \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon} | c_B + z] = 0$. *Q.E.D.*

Theorem 3 shows that if B is weakly more risk averse than A , then \tilde{c}_A is distributed as \tilde{c}_B plus a risk premium plus random noise. The distributions of the risk premium and the noise term are typically not uniquely determined. Also, it is possible that the weakly less risk averse agent's payoff can have a higher mean and a lower variance than the weakly more risk averse agent's payoff as we will see in example VI.2. This can happen because although adding condition-mean-zero noise always increases variance, adding the non-negative random variable decreases variance if it is sufficiently negatively correlated with the rest (Since $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) + Var(\tilde{z}) + 2Cov(\tilde{c}_B, \tilde{z})$, if $Cov(\tilde{c}_B, \tilde{z}) < -\frac{1}{2}(Var(\tilde{z}) + Var(\tilde{\varepsilon}))$, then $Var(\tilde{c}_A) < Var(\tilde{c}_B)$). This should not be too surprising, given that it is well-known that in general variance is not a good measure of risk⁶ for von Neumann-Morgenstern utility functions,⁷ and for general distributions in a complete market, mean-variance preferences are hard to justify.

Our second main result says that when either of the two agents has non-increasing absolute risk aversion, we can choose \tilde{z} to be non-stochastic, in which case $z = E[\tilde{c}_A - \tilde{c}_B]$. The basic idea is as follows. If either agent has non-increasing absolute risk aversion, then we can construct a new agent A^* whose consumption equals to A 's consumption plus $E[\tilde{c}_A - \tilde{c}_B]$. We can therefore get the distributional results for agent A^* and B since A^* is weakly less risk averse than B .

Theorem 5 *If B is weakly more risk averse than A and either of the two agents has non-increasing absolute risk aversion, then \tilde{c}_A is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon} | c_B + z] = 0$.*

⁶See, for example Hanoch and Levy (1970), and the survey of Machina and Rothschild (2008).

⁷If von Neumann-Morgenstern utility functions are mean-variance preferences, then they have to be quadratic utility functions, but quadratic preferences are not appealing because they are not increasing everywhere and they have increasing risk aversion where they are increasing.

Proof: Define the utility function $U_{A^*}(\tilde{c}) = U_A(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])$. In the case when A has non-increasing absolute risk aversion, A^* is weakly less risk averse than B because A is weakly less risk averse than B and non-increasing risk aversion of A implies that A^* is weakly less risk averse than A . In the case when B has non-increasing absolute risk aversion, B^* with utility $U_{B^*} = U_B(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])$ is weakly less risk averse than B and A^* is weakly less risk averse than B^* . Therefore, in both cases, we have that A^* is weakly less risk averse than B .

Give agent A^* initial wealth $w_{A^*} = w_0 - E[\tilde{\rho}]E[\tilde{c}_A - \tilde{c}_B]$, where w_0 is the common initial wealth of agent A and B . A^* 's problem is

$$\begin{aligned} \max_{\tilde{c}} E[U_A(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])], \\ \text{s.t. } E[\tilde{\rho}\tilde{c}] = w_{A^*}. \end{aligned} \quad (10)$$

The first order conditions are related to the optimality of \tilde{c}_A for agent A . To satisfy the budget constraints, agent A^* will optimally hold $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$.

By Lemma 1, $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ and \tilde{c}_B are monotonely related and there is a critical value c^* above which $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ is weakly larger and below which \tilde{c}_B is weakly larger. This implies that

$$(V'(-\tilde{c}_B) - V'(-c^*))(\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] - \tilde{c}_B) \geq 0, \quad (11)$$

where $V(\cdot)$ is an arbitrary concave function and $V'(\cdot)$ is any selection from the sub-gradient correspondence $\nabla V(\cdot)$. The concavity of $V(\cdot)$ implies that

$$V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]) - V(-\tilde{c}_B) \leq V'(-\tilde{c}_B)(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B] + \tilde{c}_B). \quad (12)$$

(11) and (12) imply that

$$V(-\tilde{c}_B) - V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]) \geq V'(-c^*)(\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] - \tilde{c}_B). \quad (13)$$

We have

$$E[V(-\tilde{c}_B) - V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B])] \geq E[V'(-c^*)(\tilde{c}_A - \tilde{c}_B - E[\tilde{c}_A - \tilde{c}_B])] = 0. \quad (14)$$

Therefore, for any concave function $V(\cdot)$, we have

$$E[V(-\tilde{c}_B)] \geq E[V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B])]. \quad (15)$$

By Theorem 2, this says that $-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]$ is distributed as $-\tilde{c}_B + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon} - c_B] = 0$. This is exactly the same as saying that $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ is distributed as $\tilde{c}_B + (-\tilde{\varepsilon})$, where $E[-\tilde{\varepsilon}|c_B] = 0$. Relabel $-\tilde{\varepsilon}$ as $\tilde{\varepsilon}$, and we have

$$\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] \sim \tilde{c}_B + \tilde{\varepsilon}, \text{ i.e., } \tilde{c}_A \sim \tilde{c}_B + E[\tilde{c}_A - \tilde{c}_B] + \tilde{\varepsilon}, \quad (16)$$

where $E[\tilde{\varepsilon}|c_B + z] = 0$.

Q.E.D.

The non-increasing absolute risk aversion condition is sufficient but not necessary. A quadratic utility function has increasing absolute risk aversion. But, as illustrated by example VI.1, the non-negative random variable can still be chosen to be a constant for quadratic utility functions (which can be viewed as an implication of two-fund separation and Theorem 7). If the non-negative random variable can be chosen to be a constant, then we have the following Corollary:

Corollary 1 *If B is weakly more risk averse than A and either of the two agents has non-increasing absolute risk aversion, then $Var(\tilde{c}_A) \geq Var(\tilde{c}_B)$.*

Proof: From Theorem 5, the non-negative random variable \tilde{z} can be chosen to be the constant $E[\tilde{c}_A - \tilde{c}_B]$. Then we have $E(\tilde{\varepsilon}|\tilde{c}_B) = 0$, which implies that $Cov(\tilde{\varepsilon}, \tilde{c}_B) = 0$. Therefore, $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) + 2Cov(\tilde{c}_B, \tilde{\varepsilon}) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) \geq Var(\tilde{c}_B)$. Q.E.D.

IV. Extension to a Multiple-Period Model

We now examine our main results in a multiple period model. We assume that each agent's problem is:

Problem 2

$$\begin{aligned} \max_{\tilde{c}_t} E\left[\sum_{t=1}^T D_t U_i(\tilde{c}_t)\right], \\ s.t. \quad E\left[\sum_{t=1}^T \tilde{\rho}_t \tilde{c}_t\right] = w_0, \end{aligned} \tag{17}$$

where $i = A$ or B indexes the agent, D_t is a discount factor (e.g., $D_t = e^{-\kappa t}$ if the pure rate of time discount κ is constant), and $\tilde{\rho}_t$ is the state price density in period t . Again, we will assume that both agents have optimal random consumptions, denoted \tilde{c}_{At} and \tilde{c}_{Bt} , and both \tilde{c}_{At} and \tilde{c}_{Bt} have finite means. The first order condition gives us

$$U'_i(\tilde{c}_{it}) = \lambda_i \frac{\tilde{\rho}_t}{D_t}, i = A, B,$$

we have

$$\tilde{c}_{it} = I_i \left(\lambda_i \frac{\tilde{\rho}_t}{D_t} \right),$$

where $I_i(\cdot)$ is the inverse function of $U'_i(\cdot)$, by negativity of the second order derivatives, \tilde{c}_{it} is a decreasing function of $\tilde{\rho}_t$. By similar arguments in the one period model, we have

Lemma 2 *If B is weakly more risk averse than A , then*

1. *there exists some critical consumption level c_t^* (can be $\pm\infty$) such that $\tilde{c}_{At} \geq \tilde{c}_{Bt}$ when $\tilde{c}_{Bt} \geq c_t^*$, and such that $\tilde{c}_{At} \leq \tilde{c}_{Bt}$ when $\tilde{c}_{Bt} \leq c_t^*$;*
2. *if it happens that the budget shares as a function of time are the same for both agents at some time t , i.e., $E[\tilde{\rho}_t \tilde{c}_{At}] = E[\tilde{\rho}_t \tilde{c}_{Bt}]$, then $E[\tilde{c}_{At}] \geq E[\tilde{c}_{Bt}]$, and we have $\tilde{c}_{At} \sim \tilde{c}_{Bt} + \tilde{z}_t + \tilde{\varepsilon}_t$, where $\tilde{z}_t \geq 0$ and $E[\tilde{\varepsilon}_t | c_{Bt} + z_t] = 0$. And if $\tilde{c}_{At} \neq \tilde{c}_{Bt}$, then neither \tilde{z}_t nor $\tilde{\varepsilon}_t$ is identically zero. In particular, if the budget shares are the same for all t , then this distributional condition holds for all t .*

The proof of Lemma 2 is the same as that in the one-period model (Lemma 1, and Theorem 3). If the D_t is not the same for both agents, or the same for the two agents without any restriction on budget shares, then the distributional condition may not hold in any period. For example, if the weakly more risk averse agent B spends most of the money earlier but the weakly less risk averse agent A spends more later, then the mean payoff could be higher in an earlier period for the weakly more risk averse agent, i.e., $E[\tilde{c}_{Bt}] > E[\tilde{c}_{At}]$.

Now, assume both agents have the same discount factor D_t and choose the period and consumption using a mixture model: first choose t with probability $\mu_t = \frac{D_t}{\sum_{t=1}^T D_t}$, and then choose $\tilde{\rho}_t$ from its distribution. Then, we will show that, under this probability measure $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$.

Definition 1 *Suppose the original probability space has probability measure P over states Ω with filtration $\{\mathcal{F}_t\}$. We define the discrete random variable τ on associated*

probability space $(\Omega^*, \mathcal{F}^*, P^*)$ so that $P^*(\tau = t) = \mu_t \equiv D_t / (\sum_{t=1}^T D_t)$. We then define a single-period problem on a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Define the state of nature in the product space $(t, \omega) \in \hat{\Omega} \equiv \Omega^* \times \Omega$ with t and ω drawn independently. Let $\hat{\mathcal{F}}$ be the optional σ -algebra, which is the completion of $\mathcal{F}^* \times \mathcal{F}_\tau$. The synthetic probability measure is the one consistent with independence generated from $\hat{P}(f^*, f) = P^*(f^*) \times P(f)$ for all subsets $f^* \in \mathcal{F}^*$ and subsets $f \in \mathcal{F}_\tau$.

The synthetic probability measure assigns a probability measure that looks like a mixture model, drawing time first assigning probability μ_t to time t , and then drawing from $\tilde{\rho}_t$ using its distribution in the original problem.

Recall that under the original probability measure, each agent's problem is given in (17). Now we want to write down an equivalent problem, in terms of the choice of distribution of each \tilde{c}_t , but with the new synthetic probability measure. The consumption \tilde{c} under the new probability space over which synthetic probabilities are defined is a function of $\tilde{\rho}$ and t ; we identify $\tilde{c}(\tilde{\rho}, t)$ with what used to be $\tilde{c}_t(\tilde{\rho})$. To write the objective function in terms of the synthetic probabilities, we can write

$$\begin{aligned} E\left[\sum_{t=1}^T D_t U(\tilde{c}_t)\right] &= \sum_{t=1}^T D_t E[U(\tilde{c}_t)] = \sum_{t=1}^T \left(\sum_{s=1}^T D_s\right) \mu_t \hat{E}[U(\tilde{c})|t] \\ &= \left(\sum_{s=1}^T D_s\right) \sum_{t=1}^T \mu_t \hat{E}[U(\tilde{c})|t] = \left(\sum_{s=1}^T D_s\right) \hat{E}[U(\tilde{c})], \end{aligned} \quad (18)$$

where \hat{E} denotes the expectation under the synthetic probability. $\sum_{s=1}^T D_s$ is a positive constant, so the objective function is equivalent to maximizing $\hat{E}[U(\tilde{c})]$.

Now, we can write the budget constraint in terms of the synthetic probabilities,

$$w_0 = E\left[\sum_{t=1}^T \tilde{\rho}_t \tilde{c}_t\right] = \sum_{t=1}^T \mu_t E\left[\frac{\tilde{\rho}_t}{\mu_t} \tilde{c}_t\right] = \sum_{t=1}^T \mu_t \hat{E}\left[\frac{\tilde{\rho}}{\mu} \tilde{c}|t\right] = \hat{E}\left[\frac{\tilde{\rho}}{\mu} \tilde{c}\right]. \quad (19)$$

Then we can apply our single-period results (Theorem 3, 4 and 5) to derive that our main results holds on a mixture model of the \tilde{c}_A and \tilde{c}_B over time:

Theorem 6 *In a multiple-period model, assume agent A and B have the same discount factor D_t and solve Problem 2, and let \tilde{c}_A and \tilde{c}_B be the optimal consumption of A and B respectively under the synthetic probability measure, we have*

1. *if B is weakly more risk averse than A, then, $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$ under the synthetic probabilities, where $\tilde{z} \geq 0$, $\hat{E}[\tilde{\varepsilon}|c_B + z] = 0$;*
2. *If for all $\tilde{\rho} \in \mathcal{P}$, $\hat{E}[\tilde{c}_A] \geq \hat{E}[\tilde{c}_B]$, then B is weakly more risk averse than A. This implies a converse result of statement 1: if for all $\tilde{\rho} \in \mathcal{P}$, \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $\hat{E}[\tilde{\varepsilon}|c_B + z] = 0$, then B is weakly more risk averse than A;*
3. *If B is weakly more risk averse than A and either of the two agents has non-increasing absolute risk aversion, then \tilde{c}_A is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = \hat{E}[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $\hat{E}[\tilde{\varepsilon}|c_B + z] = 0$.*

Therefore, if the budget shares are not the same for both agents at each time period t , then the distributional result may not hold period-by-period in a multiple-period model with time-separable von Neumann-Morgenstern utility having identical weights over time. However, Theorem 6 implies that our main results still hold under the synthetic probabilities in a multiple-period model. This results retain the spirit of our main results while acknowledging that changing risk aversion may cause consumption to shift over time.

V. Possibly Incomplete Market Case

Our result still holds in a two-asset world with a risk-free asset. For a two-asset world without a risk-free asset, we have a counter-example to our result holding. Therefore, our main result does not hold in general with incomplete markets. However, our result holds in a two-risky-asset world if we make enough assumptions about asset payoffs and the risk-aversion measure. Also, each two-asset result has a natural analog for models with many assets and two-fund separation, since the portfolio payoffs will be the same as in a two-asset model in which only the two funds are traded.⁸ Note that while this section is intended to ask to what extent our results can be extended to incomplete markets, the results also apply to complete markets with two-fund separation.

First, we show that our main result still holds in a two-asset world with a risk-free asset. The proof is in two parts. The first part is the standard result: decreasing the risk aversion increases the portfolio allocation to the asset with higher return. The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. To show the second part, we use the following Lemma:

Lemma 3 1. If $E[\tilde{q}] = 0$ and $0 \leq m_1 \leq m_2$, then $m_2\tilde{q} \sim m_1\tilde{q} + \tilde{\varepsilon}$, where

$$E[\tilde{\varepsilon}|m_1q] = 0.$$

2. Let $E[\tilde{\chi}]$ be finite, $E[\tilde{q}|\chi] \geq 0$, and $0 \leq m_1 \leq m_2$. Then $\tilde{\chi} + m_2\tilde{q} \sim \tilde{\chi} + m_1\tilde{q} + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = (m_2 - m_1)E[\tilde{q}|\chi] \geq 0$ and $E[\tilde{\varepsilon}|\chi + m_1q + z] = 0$.

Proof: We prove 2, and 1 follows immediately by setting $\tilde{\chi} = 0$ and $E[\tilde{q}] = 0$. Let

⁸See Cass and Stiglitz (1970) and Ross (1978) for characterization of two-fund separation, *i.e.*, for portfolio choice to be equivalent to choice between two mutual funds of assets.

$\tilde{z}_0 \equiv E[\tilde{q}|\chi]$ and $\tilde{z} \equiv (m_2 - m_1)\tilde{z}_0$. By Theorem 2, we only need to show that, for any concave function $V(\cdot)$, $E[V(\tilde{\chi} + m_2\tilde{q})] \leq E[V(\tilde{\chi} + m_1\tilde{q} + \tilde{z})]$. Fix $V(\cdot)$ and let $V'(\cdot)$ be any selection from its subgradient correspondence $\nabla V(\cdot)$ (so $V'(\cdot)$ is the derivative of $V(\cdot)$ whenever it exists). The concavity of $V(\cdot)$ and the definitions of \tilde{z}_0 and \tilde{z} imply that

$$V(\tilde{\chi} + m_2\tilde{q}) - V(\tilde{\chi} + m_1\tilde{q} + \tilde{z}) \leq V'(\tilde{\chi} + m_1\tilde{q} + \tilde{z})(m_2 - m_1)(\tilde{q} - \tilde{z}_0). \quad (20)$$

Furthermore, $V'(\cdot)$ nonincreasing, $m_2 \geq m_1 \geq 0$, and the definitions of \tilde{z}_0 and \tilde{z} imply

$$(V'(\tilde{\chi} + m_1\tilde{q} + \tilde{z}) - V'(\tilde{\chi} + m_2\tilde{z}_0))(m_2 - m_1)(\tilde{q} - \tilde{z}_0) \leq 0. \quad (21)$$

From (20), (21), and the definitions of \tilde{z}_0 and \tilde{z} , we get

$$\begin{aligned} E[V(\tilde{\chi} + m_2\tilde{q})] - E[V(\tilde{\chi} + m_1\tilde{q} + \tilde{z})] &\leq E[V'(\tilde{\chi} + m_1\tilde{q} + \tilde{z})(m_2 - m_1)(\tilde{q} - \tilde{z}_0)] \\ &\leq E[V'(\tilde{\chi} + m_2\tilde{z}_0)(m_2 - m_1)(\tilde{q} - \tilde{z}_0)] \\ &= E[E[V'(\tilde{\chi} + m_2\tilde{z}_0)(m_2 - m_1)(\tilde{q} - \tilde{z}_0)|\chi]] = 0. \end{aligned}$$

Q.E.D.

Now, we consider the following portfolio choice problem:

Problem 3 (*Possibly Incomplete Market with Two Assets*) Agent i 's ($i = A, B$) problem is

$$\max_{\alpha_i \in \mathbb{R}} E[U_i(w_0\tilde{x} + \alpha_i w_0\tilde{v})],$$

where w_0 is the initial wealth, α_i is the proportion invested in the second asset, and

\tilde{v} is the excess of the return on the second asset over the first asset, i.e., $\tilde{v} = \tilde{y} - \tilde{x}$, where \tilde{x} and \tilde{y} are the total returns on the two assets. We assume that $E[\tilde{v}] \geq 0$, \tilde{v} is nonconstant, and $E[\tilde{v}]$ and $E[\tilde{x}]$ are finite.

We denote agent A and B 's respective optimal investments in the risky asset with payoff \tilde{y} by α_A^* and α_B^* . The payoff for agent A is $\tilde{c}_A = w_0\tilde{x} + \alpha_A^*w_0\tilde{v}$ and agent B 's payoff is $\tilde{c}_B = w_0\tilde{x} + \alpha_B^*w_0\tilde{v}$. We maintain the utility assumptions made earlier: $U_i'(\cdot) > 0$ and $U_i''(\cdot) < 0$, so \tilde{v} nonconstant implies that α_A^* and α_B^* are unique if they exist. We have the following well-known result.

Lemma 4 *Suppose \tilde{x} is riskless (\tilde{x} nonstochastic), if B is weakly more risk averse than A , then the agents' solutions to Problem 3 satisfy $\alpha_A^* \geq \alpha_B^*$.*

Proof: The first-order condition of A 's problem is:

$$E[U_A'(xw_0 + \alpha_A^*w_0\tilde{v})w_0\tilde{v}] = 0. \quad (22)$$

The analogous expression for B is $\varphi(\alpha_B^*) = 0$, where

$$\varphi(\alpha_B^*) \equiv E[U_B'(xw_0 + \alpha_B^*w_0\tilde{v})w_0\tilde{v}]. \quad (23)$$

Since $U_B(\cdot) = G(U_A(\cdot))$, where $G'(\cdot) > 0$ and $G''(\cdot) \leq 0$, we have:

$$\varphi(\alpha_A^*) = E[G'(U_A(xw_0 + \alpha_A^*w_0\tilde{v}))U_A'(xw_0 + \alpha_A^*w_0\tilde{v})w_0\tilde{v}] \leq 0. \quad (24)$$

Here is the proof of the last inequality. The expectation in (24) is the same as that in (22) except that marginal utility $U_A'(\cdot)$ is multiplied by $G'(\cdot)$, a positive decreasing function of \tilde{v} . Therefore, (22) underweights the positive values of

$E[U'_A(xw_0 + \alpha_A^* w_0 \tilde{v})w_0 \tilde{v}]$, which obtain for $\tilde{v} > 0$ relative to the negative values of $E[U'_A(xw_0 + \alpha_A^* w_0 \tilde{v})w_0 \tilde{v}]$, which obtain for $\tilde{v} < 0$. Because the expectation in (22) is zero, the expectation in (24) must be a non-positive number. Finally, the concavity of $U_B(\cdot)$ implies that $\varphi(\cdot)$ is decreasing, and therefore from (23) and (24), we must have $\alpha_A^* \geq \alpha_B^*$. Q.E.D.

Lemma 4 implies that decreasing the risk aversion increases the portfolio allocation to the asset with higher return. Now, we show that our main result still holds in a two-asset world with a risk-free asset. We have

Theorem 7 (*Two-asset World with a Riskless Asset*) *Consider the two-asset world with a riskless asset (\tilde{x} nonstochastic) of Problem 3, if B is weakly more risk averse than A in the sense of Arrow and Pratt, then \tilde{c}_A is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.*

Proof: When the first asset in Problem 3 is riskless, then we have $\tilde{c}_A - E[\tilde{c}_A] = \alpha_A^* w_0 (\tilde{y} - E[\tilde{y}])$ and $\tilde{c}_B - E[\tilde{c}_B] = \alpha_B^* w_0 (\tilde{y} - E[\tilde{y}])$. From Lemma 4, $\alpha_A^* \geq \alpha_B^*$. Let $\tilde{q} \equiv \tilde{y} - E[\tilde{y}]$, $m_1 \equiv \alpha_B^* w_0$ and $m_2 \equiv \alpha_A^* w_0$ in the first part of Lemma 3, we have $\tilde{c}_A - E[\tilde{c}_A] \sim \tilde{c}_B - E[\tilde{c}_B] + \tilde{\varepsilon}$, which implies that \tilde{c}_A is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. Q.E.D.

Theorem 7 generalizes in obvious ways to settings with two-fund separation since optimal consumption is the same as it would be with ordering the two funds as assets. The main requirement is that one of the funds can be chosen to be riskless, for example, in a mean-variance world with a riskless asset and normal returns for risky assets.⁹ In this example, if B is weakly more risk averse than A , Theorem 7

⁹This example is a special case of two-fund separation in mean-variance worlds or the separating

tells us that $\tilde{c}_A \sim \tilde{c}_B + z + \tilde{\varepsilon}$, where $z \geq 0$ is constant and $E[\tilde{\varepsilon}|c_B + z] = 0$. We know that A 's optimal portfolio is further up the frontier than B 's, *i.e.*, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$ and $Var[\tilde{c}_A] \geq Var[\tilde{c}_B]$. This result is verified by noting that we can choose $z = E[\tilde{c}_A - \tilde{c}_B]$, $\tilde{\varepsilon} \sim N(0, Var[\tilde{c}_A] - Var[\tilde{c}_B])$, and $\tilde{\varepsilon}$ is drawn independently of \tilde{c}_B .

Now, we examine the case with two risky assets in Problem 3. For a two-asset world without a riskless asset, we have a counter-example to our result holding. In the counter-example, $\alpha_A^* > \alpha_B^*$, but the distributional result does not hold.

Example V.1 We assume that there are two risky assets and four states. The probabilities for the four states are 0.2, 0.3, 0.3 and 0.2 respectively. The payoff of \tilde{x} is $(10 \ 8 \ 1 \ 1)^T$ and the net payoff \tilde{v} is $(-1 \ 1 \ 1 \ -1)^T$. Agent's utility function is $U_i(\tilde{w}_i) = -e^{-\delta_i \tilde{w}_i}$, where $i = A, B$, and \tilde{w}_i is agent i 's terminal wealth. We assume that agent B is weakly more risk averse than A with $\delta_A = 1$ and $\delta_B = 1.5$. The agents solve Problem 3 with initial wealth $w_0 = 1$.

The agents' problems are:

$$\max_{\alpha_A} 0.2e^{-(10-\alpha_A)} + 0.3e^{-(8+\alpha_A)} + 0.3e^{-(1+\alpha_A)} + 0.2e^{-(1-\alpha_A)},$$

and

$$\max_{\alpha_B} 0.2e^{-1.5(10-\alpha_B)} + 0.3e^{-1.5(8+\alpha_B)} + 0.3e^{-1.5(1+\alpha_B)} + 0.2e^{-1.5(1-\alpha_B)}.$$

First-order conditions give $\alpha_A^* = \frac{1}{2} \log\left(\frac{3+3e^{-7}}{2+2e^{-9}}\right) = 0.2$, and $\alpha_B^* = \frac{1}{3} \log\left(\frac{3+3e^{-10.5}}{2+2e^{-13.5}}\right) = 0.135$. Therefore, agent A 's portfolio payoff is $(9.8 \ 8.2 \ 1.2 \ 0.8)^T$ and agent B 's portfolio payoff is $(9.865 \ 8.135 \ 1.135 \ 0.865)^T$. If agent A 's payoff $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, then we have $Pr(\tilde{\varepsilon} \geq 0|c_B + z) > 0$, therefore, we have $\max \tilde{c}_A \geq \max \tilde{c}_B$. However, in this example, we can see that $\max \tilde{c}_A = 9.8$ and $\max \tilde{c}_B = 9.865$,
distributions of Ross (1978).

i.e., $\max \tilde{c}_A < \max \tilde{c}_B$. Contradiction! Therefore, in general, our result does not hold in a two-asset world without a riskless asset. *Q.E.D.*

It is a natural question to ask whether our main result holds in a two risky asset world if we make enough assumptions about asset payoffs. We can, if we use Ross's stronger measure of risk aversion (see Ross (1981)) and his payoff distributional condition. We have

Theorem 8 (*Two Risky Assets with Ross's Measure*) *Consider the two-risky-asset world of Problem 3 with $E[\tilde{v}|x] \geq 0$ for all x . If B is weakly more risk averse than A under Ross's stronger measure of risk aversion, then \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, and $\tilde{z} \geq 0$.*

Proof: Our proof is in two parts. The first part is from Ross (1981): if agent A is weakly less risk averse than B under Ross's stronger measure, then $\alpha_A^* \geq \alpha_B^*$. The first order condition of A 's problem is

$$E[U'_A(w_0\tilde{x} + \alpha_A^*w_0\tilde{v})w_0\tilde{v}] = 0. \tag{25}$$

The analogous expression for B is $\varphi(\alpha_B^*) = 0$, where

$$\varphi(\alpha_B^*) \equiv E[U'_B(w_0\tilde{x} + \alpha_B^*w_0\tilde{v})w_0\tilde{v}]. \tag{26}$$

From Ross (1981), if B is weakly more risk averse than A under Ross's stronger measure, then there exists $\lambda > 0$ and a concave decreasing function $G(\cdot)$, such that $U_B(\cdot) = \lambda U_A(\cdot) + G(\cdot)$. Therefore,

$$\varphi(\alpha_A^*) = E[(\lambda U'_A(w_0\tilde{x} + \alpha_A^*w_0\tilde{v}) + G'(w_0\tilde{x} + \alpha_A^*w_0\tilde{v}))w_0\tilde{v}]$$

$$= E[G'(w_0\tilde{x} + \alpha_A^*w_0\tilde{v})w_0\tilde{v}] = E[E[G'(w_0\tilde{x} + \alpha_A^*w_0\tilde{v})w_0\tilde{v}|x]] \leq 0, \quad (27)$$

where the last inequality is a consequence of the fact that $G'(\cdot)$ is negative and decreasing while $E[\tilde{v}|x] \geq 0$. The concavity of $U_B(\cdot)$ implies that $\varphi(\cdot)$ is decreasing. Therefore, from (25) and (27), we have $\alpha_A^* \geq \alpha_B^*$.

The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. Let $\tilde{q} \equiv \tilde{v}$, $\tilde{\chi} \equiv w_0\tilde{x}$, $m_1 \equiv \alpha_B^*w_0$ and $m_2 \equiv \alpha_A^*w_0$ in Lemma 3, part 2, we have $w_0\tilde{x} + \alpha_A^*w_0\tilde{v} \sim w_0\tilde{x} + \alpha_B^*w_0\tilde{v} + \tilde{z} + \tilde{\varepsilon}$, *i.e.*, $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = w_0(\alpha_A^* - \alpha_B^*)E[\tilde{v}|x] \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. *Q.E.D.*

Theorem 8 implies that our main result holds when we use Ross's stronger measure of risk aversion with the assumption of $E[\tilde{v}|x] \geq 0$. If the condition $E[\tilde{v}|x] \geq 0$ is not satisfied, then our main result may not hold even when we use Ross's stronger measure of risk aversion as we can see in the following example.

Example V.2 We assume that there are two risky assets and four states. The probabilities for the four states are 0.3, 0.2, 0.3 and 0.2 respectively. The payoff of \tilde{x} is $(10 \ 8 \ 1 \ 1)^T$ and the net payoff \tilde{v} is $(-1 \ 1 \ 1 \ -1)^T$. Agent A 's utility function is $U_A(\tilde{w}_A) = e^6\tilde{w}_A - e^{-\tilde{w}_A}$, and agent B 's utility function is $U_B(\tilde{w}_B) = \tilde{w}_B - e^{6-1.5\tilde{w}_B}$, where \tilde{w}_i is the terminal wealth of agent i . The agents solve Problem 3 with initial wealth $w_0 = 1$. We have

$$\frac{U_B''(w)}{U_A''(w)} = \frac{2.25e^{6-1.5w}}{e^{-w}} = 2.25e^{6-0.5w}, \quad \frac{U_B'(w)}{U_A'(w)} = \frac{1 + 1.5e^{6-1.5w}}{e^6 + e^{-w}}.$$

Therefore, $\inf_w \frac{U_B''(w)}{U_A''(w)} > \sup_w \frac{U_B'(w)}{U_A'(w)}$, for any $0 \leq w \leq 10$, which implies that agent B is strictly more risk aversion than agent A under Ross's stronger measure of risk

aversion.

The Agents' problems are:

$$\begin{aligned} \max_{\alpha_A} & 0.3 \left(e^6(10 - \alpha_A) - e^{-(10-\alpha_A)} \right) + 0.2 \left(e^6(8 + \alpha_A) - e^{-(8+\alpha_A)} \right) \\ & + 0.3 \left(e^6(1 + \alpha_A) - e^{-(1+\alpha_A)} \right) + 0.2 \left(e^6(1 - \alpha_A) - e^{-(1-\alpha_A)} \right), \end{aligned}$$

and

$$\begin{aligned} \max_{\alpha_B} & 0.3 \left(10 - \alpha_B - e^{6-1.5(10-\alpha_B)} \right) + 0.2 \left(8 + \alpha_B - e^{6-1.5(8+\alpha_B)} \right) \\ & + 0.3 \left(1 + \alpha_B - e^{6-1.5(1+\alpha_B)} \right) + 0.2 \left(1 - \alpha_B - e^{6-1.5(1-\alpha_B)} \right). \end{aligned}$$

From the first order condition, $e^{2\alpha_A^*} = \frac{3+2e^{-7}}{2+3e^{-9}}$, *i.e.*, $\alpha_A^* = \frac{1}{2} \log \left(\frac{3+2e^{-7}}{2+3e^{-9}} \right) = 0.2029$, and $e^{3\alpha_B^*} = \frac{3e^{-1.5}+2e^{-12}}{2e^{-1.5}+3e^{-15}}$, *i.e.*, $\alpha_B^* = \frac{1}{3} \log \left(\frac{3e^{-1.5}+2e^{-12}}{2e^{-1.5}+3e^{-15}} \right) = 0.1352$. Therefore, agent A 's portfolio payoff is $(9.7971 \ 8.2029 \ 1.2029 \ 0.7971)^T$ and agent B 's portfolio payoff is $(9.8648 \ 8.1352 \ 1.1352 \ 0.8648)^T$. If agent A 's payoff $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, then we have $Pr(\tilde{\varepsilon} \geq 0|c_B + z) > 0$. Therefore, we have $\max \tilde{c}_A \geq \max \tilde{c}_B$. However, in this example, we can see that $\max \tilde{c}_A = 9.7971$ and $\max \tilde{c}_B = 9.8648$, *i.e.*, $\max \tilde{c}_A < \max \tilde{c}_B$. Contradiction! Therefore, in a two-risky asset world, our main result does not hold in general even under Ross's stronger measure of risk aversion if we don't make the assumption that $E[\tilde{v}|x] \geq 0$. *Q.E.D.*

An alternative to the approach following Ross (1981) is the approach of Kihlstrom, Romer and Williams (1981) for handling random base wealth. They show that the Arrow-Pratt measure works if we restrict attention to comparisons in which (1) at least one of the utility functions has nonincreasing absolute risk aversion and (2) base wealth is independent of the other gambles. Here is how their argument works.

The independence implies that we can convert a problem with random base wealth x to a problem with nonrandom base wealth by using the indirect utility functions $\hat{U}_i(w) \equiv E[U_i(\tilde{x} + w)]$, and our results for nonrandom base wealth apply directly. For this to work, the indirect utility functions \hat{U}_A and \hat{U}_B must inherit the risk aversion ordering from U_A and U_B , which as they point out, does not happen in general. However, letting $F(\cdot)$ be the distribution function of \tilde{x} , simple calculations tell us that provided integrals exist, we can write

$$-\frac{\hat{U}_i''(w)}{\hat{U}_i'(w)} = \int \frac{U_i'(\tilde{x} + w)}{\int U_i'(\tilde{y} + w)dF(\tilde{y} + w)} \left(-\frac{U_i''(\tilde{x} + w)}{U_i'(\tilde{x} + w)} \right) dF(\tilde{x}) \quad (28)$$

For both agents, the risk aversion of the indirect utility function is therefore a weighted average of the risk aversion of the direct utility function, but the weights are different so the risk aversion ordering is not preserved in general (since the more risk averse agent may have relatively higher weights from wealth regions where both agents have small risk aversion). However, we do know that the more risk averse agents' weights put relatively higher weight on lower wealth levels (since i 's absolute risk aversion is $-d \log(U_i'(w)/dw)$), so if either agent has nonincreasing absolute risk aversion, then the risk aversion ordering of the direct utility function is inherited by the indirect utility function. Subject to existence of some integrals (ensured by compactness in their paper), their results and our Theorem 7 imply that if B is weakly more risk averse than A, at least one of U_A and U_B has nonincreasing absolute risk aversion, and \tilde{v} is independent of \tilde{x} , then our main result holds: $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

As we have shown that our main result does not hold in general in the traditional type of incomplete markets where portfolio payoffs are restricted to a subspace. How-

ever, it is an open question whether the results extend to more interesting models of incomplete markets in which there is a reason for the incompleteness. For example, a market that is complete over states distinguished by security returns and incomplete over other private states (see Dybvig (1992) or Chen and Dybvig (2009)). Another type of incompleteness comes from a nonnegative wealth constraint (which is an imperfect solution to information problems when investors have private information or choices related to default), which means agents have individual incompleteness and cannot fully hedge future non-traded wealth or else they would violate the nonnegative wealth constraint (see Dybvig and Liu (2009)).

VI. Examples

In example VI.1, we illustrate our main result with specific distribution of \tilde{c}_A , \tilde{c}_B and $\tilde{\varepsilon}$. In this example, the nonnegative random variable \tilde{z} can be chosen to be a constant, and therefore from Corollary 1 in Section III, the variance of the less risk averse agent's payoff is higher.

Example VI.1 B is weakly more risk averse than A , A and B have the same initial wealth $w_0 = 1$ and the utility functions are as follows

$$U_A(\tilde{c}) = -\frac{1}{2}(4 - \tilde{c})^2, \quad U_B(\tilde{c}) = -\frac{1}{2}(3 - \tilde{c})^2,$$

where $\tilde{c} < 4$ for agent A , and $\tilde{c} < 3$ for agent B . We assume that the state price density $\tilde{\rho}$ is uniformly distributed in $[0, 1]$. The first-order conditions give us $\tilde{c}_A = 4 - \lambda_A \tilde{\rho}$, and $\tilde{c}_B = 3 - \lambda_B \tilde{\rho}$. Because $E[\tilde{\rho}] = \frac{1}{2}$ and $E[\tilde{\rho}^2] = \frac{1}{3}$, the budget constraint $E[\tilde{\rho}\tilde{c}_i] = 1$, $i = A, B$, implies that $\lambda_A = 3$ and $\lambda_B = \frac{3}{2}$. Therefore, \tilde{c}_A is uniformly distributed in $[1, 4]$ and \tilde{c}_B is uniformly distributed in $[\frac{3}{2}, 3]$. We have $E[\tilde{c}_A] - E[\tilde{c}_B] = \frac{1}{4}$. Let

$\tilde{\varepsilon}$ have a Bernoulli distribution drawn independently of \tilde{c}_B with two equally possible outcomes $\frac{3}{4}$ and $-\frac{3}{4}$. It is not difficult to see that \tilde{c}_A is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = E[\tilde{c}_A] - E[\tilde{c}_B] = \frac{1}{4}$, and $\tilde{\varepsilon}$ is independent of \tilde{c}_B , which implies $E[\tilde{\varepsilon}|c_B + z] = 0$.

Next, in example VI.2, we show that in general \tilde{z} may not be chosen to be a constant. Interestingly, the variance of the weakly less risk averse agent's payoff can be smaller than the variance of the weakly more risk averse agent's payoff.

Example VI.2 B is weakly more risk averse than A , A and B have the same initial wealth $w_0 = 1$ and the utility functions are as follows

$$U_A(\tilde{c}) = -\frac{(8 - \tilde{c})^3}{3}, \quad U_B(\tilde{c}) = -\frac{(8 - \tilde{c})^5}{5},$$

where $\tilde{c} < 8$. The first-order conditions give us

$$U'_A(\tilde{c}_A) = (8 - \tilde{c}_A)^2 = \lambda_A \tilde{\rho}, \quad U'_B(\tilde{c}_B) = (8 - \tilde{c}_B)^4 = \lambda_B \tilde{\rho}. \quad (29)$$

Therefore,

$$\tilde{c}_A = 8 - \sqrt{\lambda_A \tilde{\rho}}, \quad \tilde{c}_B = 8 - (\lambda_B \tilde{\rho})^{1/4}. \quad (30)$$

From (30), we get

$$\tilde{c}_A = 8 - \sqrt{\frac{\lambda_A}{\lambda_B}} (8 - \tilde{c}_B)^2. \quad (31)$$

We have: $\tilde{c}_A \geq \tilde{c}_B$ iff $\tilde{c}_B \geq 8 - \sqrt{\frac{\lambda_B}{\lambda_A}}$. From Theorem 3, we know that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. To find an example that the variance of the less risk averse agent's payoff can be smaller, we assume that $\tilde{\rho}$ has a discrete distribution, *i.e.*, $\rho_1 = \varepsilon$ with probability $\frac{1}{2}$, $\rho_2 = \frac{1}{4}$ with probability $\frac{1}{4}$, and $\rho_3 = \frac{1}{2}$ with probability $\frac{1}{4}$. If ε is very tiny (close to zero), then from (30) and the budget constraint $E[\tilde{\rho}\tilde{c}_A] = 1$.

It is not difficult to compute $\lambda_A \approx 17.5$, $\lambda_B \approx 125.8$, $\tilde{c}_A \approx (8.591, 5.045)$ and $\tilde{c}_B \approx (8.5632, 5.184)$. Therefore, $E[\tilde{c}_A] \approx 6.73$, $E[\tilde{c}_B] \approx 6.70$, and $Var(\tilde{c}_A) \approx 1.684 < Var(\tilde{c}_B) \approx 1.704$, *i.e.*, the variance of the weakly more risk averse agent's payoff is higher. As noted in Section III, we know that if the non-negative random variable \tilde{z} can be chosen to be a constant, then $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) \geq Var(\tilde{c}_B)$. Therefore, in this example, \tilde{z} cannot be chosen to be a constant.

The next example shows that if the utility functions are not strictly concave, then our main result does not hold.

Example VI.3 B is weakly more risk averse than A , A and B have the same initial wealth $w_0 = 1$ and the utility functions are $U_A(\tilde{c}) = U_B(\tilde{c}) = \tilde{c}$. We assume there are two states with $\rho_1 = \frac{1}{2}$ with probability $\frac{1}{3}$, and $\rho_2 = \frac{1}{2}$ with probability $\frac{2}{3}$. It is not difficult to see that $\tilde{c}_A = (0, 3)$ and $\tilde{c}_B = (4, 1)$ is an optimal consumption for agent A and B for $\lambda_A = \lambda_B = 2$. We have $E[\tilde{c}_A] = E[\tilde{c}_B] = 2$ and $Var[\tilde{c}_A] = Var[\tilde{c}_B] = 2$. If $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$, then $\tilde{z} = 0$ and $\tilde{\varepsilon} = 0$, we get $\tilde{c}_A \sim \tilde{c}_B$. Contradiction! So, we cannot have $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

Example VI.3 is degenerate with constant $\tilde{\rho}$ and linear utility. It is not difficult to construct a more general example (Example VI.4), where $\tilde{\rho}$ is random and the utility function has two straight segments. The optimal portfolio is not unique on these two straight segments taken together and therefore our payoff distributional result may not hold.

Example VI.4 B is weakly more risk averse than A , A and B have the same

initial wealth $w_0 = 2$ and the utility functions are as follows

$$U_A(\tilde{c}) = U_B(\tilde{c}) = \begin{cases} -(\tilde{c} - 1)^4 + \tilde{c} & \tilde{c} < 1 \\ \tilde{c} & 1 \leq \tilde{c} \leq 2 \\ \frac{1}{256}(\tilde{c}^4 - 16\tilde{c}^3 + 72\tilde{c}^2 + 128\tilde{c} + 80) & 2 < \tilde{c} < 6 \\ \frac{1}{2}\tilde{c} + 2 & 6 \leq \tilde{c} \leq 14 \\ \frac{1}{2}e^{-(\tilde{c}-14)} - 2e^{-(\tilde{c}-14)/2} + 9 & \tilde{c} \geq 14. \end{cases}$$

In this example, the utility function has two straight segments and the optimal portfolio is not unique on these two straight segments taken together. We assume that $\rho_1 = \frac{1}{2}$ with probability $\frac{1}{2}$ and $\rho_2 = \frac{1}{4}$ with probability $\frac{1}{2}$. Then, it is not difficult to see that $\tilde{c}_A = (2, 12)$ and $\tilde{c}_B = (1, 14)$ is the optimal consumption for agent A and B for $\lambda_A = \lambda_B = 2$. So, while A is weakly less risk averse than B (their risk aversion is equal everywhere), \tilde{c}_A is not distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$ with $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

VII. Concluding Remarks

Hart (1975) proved the impossibility of deriving certain types of comparative statics results in portfolio weights. We have proven comparative statics results instead in the distribution of portfolio payoffs. Specifically, in a complete market, we show that an agent who is less risk averse than another will choose a portfolio whose payoff is distributed as the other's payoff plus a nonnegative random variable plus conditional-mean-zero noise. This result holds for any strictly concave C^2 utility functions. If either agent has non-increasing absolute risk aversion, then the non-negative random variable can be chosen to be a constant. The non-increasing absolute risk aversion condition is sufficient but not necessary. We also provide a counter example, such

that, in general, this non-negative random variable cannot be chosen to be a constant.

We further prove a converse theorem. If in all complete markets the first agent chooses a payoff that is distributed as the second's payoff, plus a non-negative random variable, plus conditional-mean-zero noise, then the first agent is less risk averse than the second agent. We also extend our main results to a multiple period model. Due to shifts in the timing of consumption, agents' optimal consumption at each date may not be ordered when risk aversion changes. However, for agents with the same pure rate of time preference, there is a natural weighting of probabilities across periods that preserves the single-period result.

The optimal consumption may not be ordered for agents with different risk aversion when agents' utility functions are concave but not strictly concave as we have shown in example VI.3 and VI.4. Intuitively, the problem is that even with identical preferences, two different optimal consumptions may not be ordered. We conjecture that there exists some canonical choice of optimal consumption for each agent that extends our main results for weakly concave preferences. Our paper derives comparative statics results in complete markets for agents with von Neumann-Morgenstern preferences. Machina (1989) has shown that many previous comparative statics results generalize to the broader class of Machina preferences (Machina (1982)). Our proofs do not generalize obviously to this class, but we conjecture that our results are still true.

We also show that our main result still holds in a two-asset world with a risk-free asset or more generally in a two-fund separation world with a risk-free asset. However, our main result is not true in general with incomplete markets. We further provide sufficient conditions under which our results still hold in a two-risky-asset world or a world with two-fund separation.

Appendix

Proof of Lemma 1: By Pratt (1964), we have the concave transform characterization¹⁰ that there exists $G(\cdot) \in C^2$, such that

$$U_B(c) = G(U_A(c)), \quad (32)$$

where $G'(\cdot) > 0$ and $G''(\cdot) \leq 0$. Using the concave transform characterization of more risk averse in (32), the first order condition (2) becomes

$$U'_A(\tilde{c}_A) = \lambda_A \tilde{\rho} = \frac{\lambda_A}{\lambda_B} \lambda_B \tilde{\rho} = \frac{\lambda_A}{\lambda_B} G'(U_A(\tilde{c}_B)) U'_A(\tilde{c}_B). \quad (33)$$

Because marginal utility is strictly decreasing, we have: if $G' < \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A > \tilde{c}_B$; if $G' = \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A = \tilde{c}_B$; and if $G' > \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A < \tilde{c}_B$. Choose c^* so that $G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$ if possible, or pick $c^* = -\infty$ if $G' < \frac{\lambda_B}{\lambda_A}$ everywhere or $c^* = +\infty$ if $G' > \frac{\lambda_B}{\lambda_A}$ everywhere. If $\tilde{c}_B \geq c^*$, then $G'(U_A(\tilde{c}_B)) \leq G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, *i.e.*, $G' \leq \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \geq \tilde{c}_B$. If $\tilde{c}_B \leq c^*$, then $G'(U_A(\tilde{c}_B)) \geq G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, *i.e.*, $G' \geq \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \leq \tilde{c}_B$. This proves statement 1.

Now suppose that A and B have equal initial wealths, then the budget constraints for the agents are that

$$E[\tilde{\rho}\tilde{c}_A] = E[\tilde{\rho}\tilde{c}_B] = w_0, \quad (34)$$

therefore, we have $E[\tilde{\rho}(\tilde{c}_A - \tilde{c}_B)] = 0$. Since $\lambda_B \tilde{\rho} = U'_B(\tilde{c}_B)$ and $U''_B < 0$, $\tilde{\rho}$ and \tilde{c}_B are negatively monotonely related. Let $\rho^* \equiv U'_B(c^*)/\lambda_B > 0$. Then $\tilde{\rho} \geq \rho^* \Rightarrow \tilde{c}_A \leq \tilde{c}_B$

¹⁰This result can be obtained by defining $G(\cdot)$ implicitly from (32) and using the implicit function theorem to compute the derivatives of $G(\cdot)$.

and $\tilde{\rho} \leq \rho^* \Rightarrow \tilde{c}_A \geq \tilde{c}_B$. Therefore, $(\tilde{\rho} - \rho^*)(\tilde{c}_A - \tilde{c}_B) \leq 0$ and we have

$$0 = E[\tilde{\rho}(\tilde{c}_A - \tilde{c}_B)] = E[\rho^*(\tilde{c}_A - \tilde{c}_B)] + E[(\tilde{\rho} - \rho^*)(\tilde{c}_A - \tilde{c}_B)] \leq \rho^* E[\tilde{c}_A - \tilde{c}_B]. \quad (35)$$

Therefore, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$. This proves statement 2.

Q.E.D.

Proof of Theorem 1: (*Sufficiency*) The monotonicity and concavity of the function and Jensen's inequality yield $E[V(\tilde{Y})] = E[V(\tilde{X} - \tilde{Z} + \tilde{\varepsilon})] = E[E[V(\tilde{X} - \tilde{Z} + \tilde{\varepsilon})|X, Z]] \leq E[V(\tilde{X} - \tilde{Z})] \leq E[V(\tilde{X})]$.

(*Necessity*) Let μ_1 be the distribution of $-\tilde{X}$, and let μ_2 be the distribution of $-\tilde{Y}$. From Theorem 9 of Strassen (1965),¹¹ the following two statements are equivalent.

- (i) For any concave nondecreasing function $V(s)$, $\int V(-s)d\mu_1(s) \geq \int V(-s)d\mu_2(s)$.
- (ii) There exists a submartingale $\tilde{\xi}_n$ ($n = 1, 2$), *i.e.*, $E[\tilde{\xi}_2|\xi_1] \geq \tilde{\xi}_1$, such that the distribution of $\tilde{\xi}_n$ is μ_n .

Let $\tilde{Z} \equiv E[\tilde{\xi}_2|\xi_1] - \tilde{\xi}_1$ and $\tilde{\varepsilon} \equiv -\tilde{\xi}_2 + E[\tilde{\xi}_2|\xi_1]$, then (ii) implies that $\tilde{Z} \geq 0$. Since $\tilde{\xi}_1 + \tilde{Z} = E[\tilde{\xi}_2|\xi_1]$, we have $E[\tilde{\varepsilon}|\xi_1 + Z] = E[(-\tilde{\xi}_2 + E[\tilde{\xi}_2|\xi_1])|E[\tilde{\xi}_2|\xi_1]] = 0$. (i) implies $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$, and since $\tilde{\xi}_2 = \tilde{\xi}_1 + (E[\tilde{\xi}_2|\xi_1] - \tilde{\xi}_1) + (\tilde{\xi}_2 - E[\tilde{\xi}_2|\xi_1])$, we have $-\tilde{Y} \sim -\tilde{X} + \tilde{Z} - \tilde{\varepsilon}$, where $\tilde{Z} \sim E[-\tilde{Y} | -X] + \tilde{X} \geq 0$ and $\tilde{\varepsilon} \sim \tilde{Y} + E[-\tilde{Y} | -X]$. It follows that $\tilde{Y} \sim \tilde{X} - \tilde{Z} + \tilde{\varepsilon}$, where $\tilde{Z} \geq 0$ and $E[\tilde{\varepsilon}|X - Z] = 0$. *Q.E.D.*

Proof of Theorem 2: The sufficiency follows directly from Jensen's inequality. The necessity can be proved using Theorem 8 in Strassen (1965). We prove it instead using Theorem 1 above. We have $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave function, and in particular, $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave nondecreasing functions. Therefore,

¹¹In applying Strassen's result, we ignore ξ_n for $n > 2$. Formally, we set $\xi_n = \xi_2$ and $\mu_n = \mu_2$ for all $n > 2$.

by Theorem 1, $\tilde{Y} \sim \tilde{X} - \tilde{Z}_1 + \tilde{\varepsilon}_1$, where $\tilde{Z}_1 \geq 0$ and $E[\tilde{\varepsilon}_1|X - Z_1] = 0$. We have

$$E[\tilde{Y}] = E[E[\tilde{Y}|X - Z_1]] = E[E[\tilde{X} - \tilde{Z}_1 + \tilde{\varepsilon}_1|X - Z_1]] = E[\tilde{X}] - E[\tilde{Z}_1] \leq E[\tilde{X}]. \quad (36)$$

Now $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave functions also implies $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave nonincreasing functions, *i.e.*, $E[V(-\tilde{X})] \geq E[V(-\tilde{Y})]$ for all concave nondecreasing functions. From Theorem 1, $-\tilde{Y} \sim -\tilde{X} - \tilde{Z}_2 + \tilde{\varepsilon}_2 \Rightarrow \tilde{Y} \sim \tilde{X} + \tilde{Z}_2 - \tilde{\varepsilon}_2$, where $\tilde{Z}_2 \geq 0$, and $E[\tilde{\varepsilon}_2|X + Z_2] = 0$. We have

$$E[\tilde{Y}] = E[E[\tilde{Y}|X + Z_2]] = E[E[\tilde{X} + \tilde{Z}_2 - \tilde{\varepsilon}_2|X + Z_2]] = E[\tilde{X}] + E[\tilde{Z}_2] \geq E[\tilde{X}]. \quad (37)$$

Therefore, $E[\tilde{X}] = E[\tilde{Y}]$, which implies $E[\tilde{Z}_1] = 0$. Since $\tilde{Z}_1 \geq 0$, we must have $\tilde{Z}_1 = 0$. It follows that $\tilde{Y} \sim \tilde{X} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|X] = 0$. *Q.E.D.*

Lemma 5 *Suppose B is not weakly more risk averse than A , then there exists an bounded nondegenerate interval $[c_1, c_2]$ and hypothetical agents A_1 and B_1 , such that A_1 strictly more risk averse than B_1 ($\forall c, -\frac{U''_{B_1}(c)}{U'_{B_1}(c)} < -\frac{U''_{A_1}(c)}{U'_{A_1}(c)}$) and $\forall c \in [c_1, c_2]$, $U_{A_1}(c) = U_A(c)$ and $U_{B_1}(c) = U_B(c)$.*

Proof of Lemma 5: If B is not weakly more risk averse than A , then there exists a constant \hat{c} , such that $-\frac{U''_B(\hat{c})}{U'_B(\hat{c})} < -\frac{U''_A(\hat{c})}{U'_A(\hat{c})}$. Since U_A and U_B are of the class of C^2 (see our assumptions in the beginning of Section II), from the continuity of $-\frac{U''_i(c)}{U'_i(c)}$, where $i = A, B$, we get that there exists an interval RA containing \hat{c} , *s.t.*, $\forall c \in RA$,

$-\frac{U_B''(c)}{U_B'(c)} < -\frac{U_A''(c)}{U_A'(c)}$. We pick $c_1, c_2 \in RA$ with $c_1 < c_2$. Now, let

$$U_{A_1}(c) = \begin{cases} a_1 - m_1 \exp\left(\frac{U_A''(c_1)}{U_A'(c_1)}c\right) & c < c_1 \\ U_A(c) & c_1 \leq c \leq c_2 \\ a_2 - m_2 \exp\left(\frac{U_A''(c_2)}{U_A'(c_2)}c\right) & c > c_2, \end{cases}$$

and let

$$U_{B_1}(c) = \begin{cases} b_1 - n_1 \exp\left(\frac{U_B''(c_1)}{U_B'(c_1)}c\right) & c < c_1 \\ U_B(c) & c_1 \leq c \leq c_2 \\ b_2 - n_2 \exp\left(\frac{U_B''(c_2)}{U_B'(c_2)}c\right) & c > c_2, \end{cases}$$

where a_j and m_j ($j = 1, 2$) are determined by the continuity and smoothness of $U_{A_1}(c)$, and b_j and n_j ($j = 1, 2$) are determined by the continuity and smoothness of $U_{B_1}(c)$. More specifically, for $j = 1, 2$, we have

$$m_j = -\frac{(U_A'(c_j))^2}{U_A''(c_j)} \exp\left(-\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right), \quad a_j = m_j \exp\left(\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right) + U_A(c_j), \quad (38)$$

and

$$n_j = -\frac{(U_B'(c_j))^2}{U_B''(c_j)} \exp\left(-\frac{U_B''(c_j)}{U_B'(c_j)}c_j\right), \quad b_j = n_j \exp\left(\frac{U_B''(c_j)}{U_B'(c_j)}c_j\right) + U_B(c_j). \quad (39)$$

Now, $U_{A_1}(c)$ is in the class of C^2 since from (38), we have:

$$-m_j \exp\left(\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right) \left(\frac{U_A''(c_j)}{U_A'(c_j)}\right)^2 = U_A''(c_j),$$

i.e., U_{A_1} is twice differentiable. Similarly, we can show that $U_{B_1}(c)$ is also in the class of C^2 . Also, we have $U_{A_1}''(c) < 0$, $U_{B_1}''(c) < 0$, and $\forall c$, $-\frac{U_{B_1}''(c)}{U_{B_1}'(c)} < -\frac{U_{A_1}''(c)}{U_{A_1}'(c)}$, *i.e.*, agent

A_1 is more risk averse than B_1 .

Q.E.D.

Lemma 6 *Suppose B is strictly more risk averse than A ($\forall c, -\frac{U''_A(c)}{U'_A(c)} < -\frac{U''_B(c)}{U'_B(c)}$), and A and B have equal initial wealths. A has an optimal choice \tilde{c}_A , and B has an optimal choice \tilde{c}_B . We assume that the state price density $\tilde{\rho}$ is not a constant. Then, we have*

1. $\tilde{c}_A \neq \tilde{c}_B$;

2. if \tilde{c}_A has a bounded support $[c_1, c_2]$, then we have $\sup \tilde{c}_A \geq \sup \tilde{c}_B$, and $\inf \tilde{c}_A \leq \inf \tilde{c}_B$.

Proof of Lemma 6: We first prove statement 1 by contradiction. If $\tilde{c}_A = \tilde{c}_B$, then we pick any two points, for example, c_3, c_4 ($c_3 < c_4$) in the support of both \tilde{c}_A and \tilde{c}_B . From the first order conditions, we get: $\frac{U'_A(c_3)}{U'_A(c_4)} = \frac{U'_B(c_3)}{U'_B(c_4)}$, i.e., $\frac{U'_A(c_3)}{U'_B(c_3)} = \frac{U'_A(c_4)}{U'_B(c_4)}$. However, from $-\frac{U''_A(c)}{U'_A(c)} < -\frac{U''_B(c)}{U'_B(c)}$, we have: $\frac{d}{dc} \left(\log \frac{U'_B(c)}{U'_A(c)} \right) < 0$, i.e., $\frac{U'_B(c)}{U'_A(c)}$ decreases in c . We have: $\frac{U'_B(c_3)}{U'_A(c_3)} > \frac{U'_B(c_4)}{U'_A(c_4)}$. Contradiction! So, $\tilde{c}_A \neq \tilde{c}_B$.

Since B is more risk averse than A , from Lemma 1, we know that there exists c^* , such that $\tilde{c}_A \geq \tilde{c}_B$ when $\tilde{c}_B \geq c^*$, and $\tilde{c}_A \leq \tilde{c}_B$ when $\tilde{c}_B \leq c^*$. And we have $c^* \in [c_1, c_2]$, or else either $\tilde{c}_A \leq \tilde{c}_B$ but $\tilde{c}_A \neq \tilde{c}_B$ or $\tilde{c}_A \geq \tilde{c}_B$ but $\tilde{c}_A \neq \tilde{c}_B$ and both could not satisfy the budget constraint ($E[\tilde{\rho}\tilde{c}_A] = E[\tilde{\rho}\tilde{c}_B] = w_0$). Therefore, \tilde{c}_A has a wider range of support than that of \tilde{c}_B . *Q.E.D.*

References

- [1] Arrow, Kenneth J. (1965) Aspects of the Theory of Risk-Bearing *Helsinki: Yrjö Jahnessonin Saatio*.
- [2] Arrow, Kenneth J. (1971) The Theory of Risk Aversion *Chapter 3 in Essays in the Theory of Risk Bearing. New York: American Elsevier, 1971*.
- [3] Cass, David, and Joseph E. Stiglitz (1970) The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds. *Journal of Economic Theory 2, 122-160*.
- [4] Dybvig, P. (1992) Hedging Nontraded Wealth: When is there Separation of Hedging and Investment? *in Hodges, S. D. (Ed) Options: Recent Advances in Theory and Practice 2, 1992, Manchester University Press*.
- [5] Dybvig P. and Chen A. (2009) Optimal Casualty Insurance, Repair, and Regulation in the Presence of a Securities Market. *Working Paper*.
- [6] Dybvig P. and Liu H. (2010) Lifetime Consumption and Investment: Retirement and Constrained Borrowing. *Journal of Economic Theory 145, 885-907*.
- [7] Gollier C. (2001) The Economics of Risk and Time. *The MIT Press*.
- [8] Hadar, J., and W.R. Russell (1969) Rules for Ordering Uncertain Prospects. *American Economic Review 59, 25-34*.
- [9] Hart, O. (1975) Some negative results on the existence of comparative static results in portfolio theory. *Review of Economic Studies, 42, 615-622*.

- [10] Hanoch, Giora, and Haim Levy (1970) Relative Effectiveness of Efficiency Criterion for Portfolio Selection. *Journal of Financial and Quantitative Analysis* 5:63-76.
- [11] Kihlstrom, Richard E., David Romer, and Steve Williams (1981) Risk Aversion with Random Initial Wealth. *Econometrica* 49, 911-920.
- [12] Machina, Mark (1982) Expected Utility Analysis without the Independence Axiom. *Econometrica*, March, 1982.
- [13] Machina, Mark (1989) Comparative Statics and Non-Expected Utility Preferences. *Journal of Economic Theory*, April, 1989.
- [14] Machina, Mark, and Michael Rothschild. (2008) Risk. *In the New Palgrave Dictionary of Economics, 2nd ed., edited by Steven N. Durlauf and Lawrence E. Blume. London: Macmillan.*
- [15] Mas-Colell, Whinston and Green (1995) Microeconomic Theory. *Oxford University Press, Inc.*
- [16] Pratt, John W. (1964) Risk Aversion in the Small and in the Large. *Econometrica* 32, 122-136.
- [17] R. Tyrrell Rockafellar (1970) Convex Analysis. *Princeton University Press.*
- [18] Ross, Stephen A. (1971) Risk and Efficiency. *Discussion Paper No. 196*
- [19] Ross, Stephen A. (1978) Mutual Fund Separation in Financial Theory-The Separating Distributions. *Journal of Economic Theory* 17, 254-286
- [20] Ross, Stephen A. (1981) Some Stronger Measures of Risk Aversion in the Small and the Large with Applications. *Econometrica* 49, 621-638.

- [21] Rothschild, M. and J.E. Stiglitz (1970) Increasing Risk: I. A. Definition. *Journal of Economic Theory* Vol.2, No.3.
- [22] Rothschild, M. and J.E. Stiglitz (1972) Addendum to “Increasing Risk: I. A. Definition”. *Journal of Economic Theory* 5, 306
- [23] V. Strassen (1965) The Existence of Probability Measures With Given Marginals. *The Annals of Mathematical Statistics*. 36, 423-439.