Endogenous Credit Constraints and Household Portfolio Choices

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Motivation

Most of previous works focus on the comparison between two cases:
With borrowing constraint vs Without borrowing constraints.

- Extant borrowing limits are numerous.
- Credit card limits and bank loans are different across households.
- Borrowing limits or Credit limits are time-varying and endogenous depending on the personal income growth and the market environment.
- However, borrowing limits in these studies are exogenously specified.
- Classifying borrowing into the aforementioned two categories without endogeneity may lead to significant loss of information.

- This paper presents a consumption-investment model for a household with Limited Commitment (without debt-repayment enforcement mechanism)
  - implications of endogenous credit limits for consumption and investment.

Literature

Literature on "Exogenous Borrowing Constraints"

- Consumption and Portfolio Selection: He and Paés (1993), Duffie et. al (1997), Koo (1998), El-Karoui and Jeanblanc (1998), Zhang (2005), Choi, Shim, and Shin (2008), Dyvbig and Liu (2010)
- Low risk-free rate: Heaton and Lucas (1996), Detemple and Serrat (2003)
- High saving rate puzzle: (Wen (2009))

Literature on Limited Commitment

- Asset Markets and Pricing: Kehoe and Levine (1993), Alvarez and Jermann (2000, 2001), Azariadis and Kass (2007), Chien and Lustig (2010), Osambela (2012), Azariadis and Choi (2012, 2013)
- International Finance Kehoe and Perri (2002, 2004)
- Firm Dynamics: Albuquerque and Hopenhayn (2004)
- Consumption Distribution: Krueger and Perri (2005)
- Political Economy: Acemoglu, Golosov and Tsyvinski (2008), Aguiar, Amardo and Gopinath (2009)

Overview: Model

- We consider a continuous-time consumption and portfolio selection problem of a household
  - who receives a stream of stochastic income
  - who can borrow up to credit constraints endogenously determined by ability to repay loan
- If the HH does not repay the loan, she will remain in an autarkic world permanently (U.S Bankruptcy, Chapter 7, and 13)
  - A household who are excluded from the market will lose the ability to intertemporally smooth her consumption and to generate a portfolio.
  - A household maximizes her utility with a sequence of dynamic constraints that is satisfied at all times.

A Few Theoretical Results and Empirical Predictions

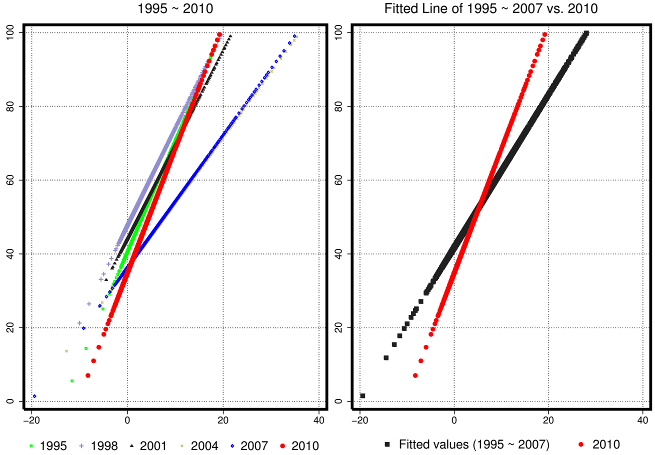
1. The household's current level of income is particularly important for determining the unsecured borrowing limit, not the current wealth level.
2. The unsecured borrowing limit is **greater** for a household with a **higher income growth rate** and **lower income volatility** (*opposite* result to that from GE macroeconomics literature of limited commitment)
3. The unsecured borrowing limit is **greater in a boom** than in a recession.

A Few More Results on the Borrowing Limit and Heterogeneous Portfolio Choice

- Limits on non-secured loans (e.g credit cards and consumer bank loans) are significantly **different across households**.
- The portfolio choice of the rich is different than that of the poor according to the changes in the Sharpe ratio
  - When the Sharpe ratio increases, **the poor reduce** the portfolio share invested in risky assets
  - When the Sharpe ratio increases, **the rich increase** the portfolio share invested in risky assets
- The heterogeneous portfolio choice over a business cycle given a fixed level of wealth,
  - The risky asset holdings are **higher for the rich** during a downturn than a boom.
  - The risky asset holdings are **lower for the poor** during a downturn than a boom.

Sharpe Ratio and Heterogeneous Portfolio Choice

Fraction of Risky Investment to Financial Assets from 1995 to 2010



Market Setup

In a complete market, a household can generate a portfolio with two securities, a riskless S_t^0 and a risky asset S_t:

- The risk-free asset:

$$\frac{dS_t^0}{S_t^0} = rdt.$$

- The risky asset (or Index):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

.

- B_t is a standard Brownian motion defined on the standard probability space $(\Omega, \mathcal{F}, P)$.
- The pricing kernel is defined by

$$H_t = \exp\left(-(r+\theta)t - \theta B_t\right).$$

with the market price of risk $\theta := (\mu - r)/\sigma$.

Income, Consumption, & Wealth

- Income stream I_t which is an $\mathcal{F}_t-$adapted predictable process.
- The present value of his future income stream $= E[\int_0^\infty H_t I_t dt]$
- H_t is the pricing kernel or the stochastic discount factor.
- Consumption rate process c_t and the portfolio process π_t are positive $\mathcal{F}_t$-predictable processes.
- Wealth process by X_t with initial endowment x evolves

$$dX_t = [rX_t + \pi_t X_t(\mu - r) - c_t + I_t]dt + \sigma \pi_t dB_t, \qquad X_0 = x. \tag{1}$$

The dynamic budget constraint can be written as the following static constraint (Karatzas and Shreve 1987 and Cox and Haung 1989)

$$\mathbb{E}\left[\int_0^\infty H_t c_t dt\right] \leq x + \mathbb{E}\left[\int_0^\infty H_t I_t dt\right]. \tag{2}$$

Endogenous Credit Constraint

- The proportion of seized income ϕ for agent's after declaring default.
- The agent's market participant constraints are

$$\mathbb{E}_t\left[\int_t^\infty e^{-\beta s}u(c_s)ds\right] \geq \mathbb{E}_t\left[\int_t^\infty e^{-\beta s}u((1-\phi)e_s)ds\right], \qquad t>0. \quad (3)$$

  - No direct restriction on borrowing or on the wealth level
  - The investor has an option to stop participating in the financial market and this option *indirectly* restrict borrowing against the future
  - Under (3) (with no information asymmetry) the agent cannot fully borrow up to the present value of his future income stream.
  - The key is to specify & analyze this time-varying and endogenous borrowing limit properly.

The economic agent wants to maximize his lifetime expected utility by selecting consumption rate c_t and portfolio amount π_t:

$$V(x) = \max_{\{c_t, \pi_t\}} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right]$$

subject to the budget constraint and endogenous credit constraint.

Lagrangian

$$\mathcal{L} \equiv \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right] + \lambda\left(x - \mathbb{E}\left[\int_0^\infty H_t(c_t - I_t)dt\right]\right)$$
$$+\mathbb{E}\left[\int_0^\infty \mathbb{E}_t\left(\int_t^\infty e^{-\beta s}\left\{u(c_s) - u((1-\phi)I_s)\right\}ds\right)dD_t\right],$$

where D_t is a non-decreasing real-valued process with $D_0 = 0$ and dD_t being the (infinitesimal) Lagrange multiplier for the participation constraint at time t. Notice that if D_t has a derivative $\dot{D}_t$, then the last term of the right hand side in (4) can be written as

$$\mathbb{E}\left[\int_0^\infty \dot{D}_t \mathbb{E}_t\left(\int_t^\infty e^{-\beta s}\left\{u(c_s) - u((1-\phi)I_s)\right\}ds\right)dt\right],$$

where $\dot{D}_t$ is the ordinary Lagrange multiplier for the constraint at t. D_t, however, is not generally differentiable. By changing the order of integration it can be written as

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\int_0^t \dot{D}_s ds\right)\left\{u(c_t) - u((1-\phi)I_t)\right\}dt\right]$$
$$= \mathbb{E}\left[\int_0^\infty e^{-\beta t} D_t\left\{u(c_t) - u((1-\phi)I_t)\right\}dt.\right]$$

Lagrangian

The Lagrangian can now be written as

$$\mathcal{L} \equiv \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right] + \lambda\left(x - \mathbb{E}\left[\int_0^\infty H_t(c_t - I_t)dt\right]\right) + \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{u(c_t) - u((1-\phi)I_t\right\} D_t dt\right]. \quad (4)$$

Let us define the remaining portion of income to consume by $\delta \equiv 1 - \phi$. Then we proceed to obtain the maximum of the Lagrangian,

$$\widetilde{V}(x;\lambda,D) = \max_{c_t}\left\{\mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right] + \lambda\left(x - \mathbb{E}\left[\int_0^\infty H_t(c_t - I_t)dt\right]\right) + \mathbb{E}\left[\int_0^\infty e^{-\beta t} D_t\left(u(c_t) - u(\delta I_t)\right)dt\right]\right\} \quad (5)$$

Lagrangian

Denoting the inverse of marginal utility $u^{'}(\cdot)$ by $L(\cdot)$, the first order condition for the optimization problem in (5) is given as

$$c_t^* = L\left(\frac{\lambda e^{\beta t} H_t}{1 + D_t}\right).$$

Notice the role of the shadow price D_t in keeping the participation constraint satisfied. When the participation constraint binds, i.e. (**??**) holds with equality, and there is a negative shock from the market, i.e., $dH_t > 0$, consumption tends to fall resulting in a violation of the participation constraint in the absence of an adjustment mechanism. The shadow price is adjusted upward ($dD_t > 0$) to prevent this from happening.

Following the convention in convex analysis, we define the convex conjugate $\tilde{u}(\cdot)$ of $u(\cdot)$ by

$$\tilde{u}(y) \equiv \max_c u(c) - yc.$$

Value Function

By the standard duality theory (see e.g., Rockafellar (1997)), the value function can be expressed as a minimum:

$$V(x) = \min_{\lambda, D_t} \widetilde{V}(x;\lambda, D) \tag{6}$$

$$= \min_{\lambda, D_t} \left\{ \mathbb{E}\left[\int_0^\infty e^{-\beta t} \left\{ (1+D_t)\tilde{u}\left(\frac{\lambda e^{\beta t} H_t}{1+D_t} \right) - D_t u(\delta I_t) \right\} dt \right] + \lambda \mathbb{E}\left[\int_0^\infty H_t I_t dt \right] + \lambda x \right\}. \tag{7}$$

Result

Theorem

The household's optimal consumption rate is given by

$$c_t^* = L\left(\frac{\lambda^* e^{\beta t} H_t}{1 + D_t^*}\right),$$

where λ^* *and* D_t^* *are optimal solutions to the minimization problem in* (7)*.*

Optimal Stopping Problems

We transform the problem into a series of optimal stopping problems. El Karoui and Jeanblanc-Piqué (1998), Dybvig and Rogers (2013), and Dybvig, Jang, and Koo (2014) have used a similar transformation. The idea is to notice the equivalence of the following, (i) for every $t > 0$, the choice of D_t in (7), and (ii) for every $v > 0$, the choice of the first time, τ_v, for D_t to reach v. Specifically, choosing the optimal shadow price for each time is equivalent to choosing the optimal time when the shadow price first reaches v for every $v > 0$.

Optimal Stopping Problems

Lemma

The household's value function can be written as the following:

$$V(x) = \inf_{\{\lambda\}} \left\{ \int_0^\infty \mathbb{E} \inf_{\tau_v} \left[\mathbb{E}_v \left[\int_{\tau_v}^\infty e^{-\beta t} \left(u \left(L \left(\frac{\lambda e^{\beta t} H_t}{1+v} \right) \right) - u(\delta I_t) \right) dt \right] \right] dv + J(\lambda) \right.$$

where the second term in the right-hand side is given by

$$J(\lambda) := \mathbb{E} \left[\int_0^\infty e^{-\beta t} \tilde{u}(\lambda e^{\beta t} H_t) dt \right] + \lambda \mathbb{E} \left[\int_0^\infty H_t I_t dt \right]$$

and τ_v is the stopping time defined by

$$\tau_v = \inf\{ s \mid D_s \leq v \}.$$

Result

Theorem

If the felicity function has constant relative risk aversion, then the value function is given by

$$V(x) = \inf_{\lambda} \left\{ \psi(\lambda; \bar{I}) + J(\lambda; \bar{I}) + \lambda x \right\}, \quad (8)$$

where the functions $\psi(\lambda; \bar{I})$ *and* $J(\lambda; \bar{I})$ *are given by*

$$\psi(\lambda; \bar{I}) = \frac{\delta^{1-\gamma} \bar{I}}{\hat{\beta}(1-\gamma+\alpha_+\gamma)} \left\{ \left(\frac{K\alpha_+ \delta^{1-\gamma}\gamma}{(1-\gamma+\alpha_+\gamma)\hat{\beta}} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\alpha_+ \lambda}{1-\alpha_+} \right) + 1 \right\},$$

$$J(\lambda; \bar{I}) = \frac{\gamma}{K(1-\gamma)} (\lambda)^{-\frac{1-\gamma}{\gamma}} + \frac{\bar{I}}{r_I} \lambda,$$

and $r_I = r - \mu_I + \theta \sigma_I$*.*
In this case the optimal λ^* *is given by*

$$\lambda^* = K^{-\gamma} \left(x + \bar{I} \left(\frac{1}{r_I} + M \right) \right)^{-\gamma},$$

with

Result

In this case the optimal λ^* is given by

$$\lambda^* = K^{-\gamma}\left(x + \bar{I}\left(\frac{1}{r_I} + M\right)\right)^{-\gamma},$$

with

$$M \equiv \frac{\alpha_+\delta}{(1-\alpha_+)\hat{\beta}(1-\gamma+\alpha_+\gamma)}\left(\frac{K\alpha_+\gamma}{(1-\gamma+\alpha_+\gamma)\hat{\beta}}\right)^{\frac{\gamma}{1-\gamma}}.$$

Furthermore, the optimal stopping time τ_v takes the following form

$$\tau_v \equiv \inf\{t > 0 : y_t \geq \mathcal{Y}_v\}, \tag{9}$$

where $y_t \equiv \lambda e^{\beta t} H_t I_t^{\gamma}$ and

$$\mathcal{Y}_v \equiv \left(\frac{K\alpha_+\delta^{1-\gamma}\gamma}{(1-\gamma+\alpha_+\gamma)\hat{\beta}}\right)^{-\frac{\gamma}{1-\gamma}}(1+v).$$

Finally, the shadow price D_t is determined as follows:

$$D_t = \max\left(0, \max_{0\leq s\leq t}\left(\frac{K\alpha_+\delta^{1-\gamma}\gamma}{(1-\gamma+\alpha_+\gamma)\hat{\beta}}\right)^{\frac{\gamma}{1-\gamma}} \cdot y_s - 1\right).$$

A Credit Market Game and A Unique Pareto-optimal Equilibrium

Game: The creditors decide a credit limit and the interest rate for the household at each point of time. The household decides the amount of borrowing and whether to default or not on the existing debt at each point of time.

Theorem

There is a unique Pareto-optimal Markov perfect equilibrium. In the unique equilibrium the household's credit limit, $\bar{L}_t$, at time t is determined by

$$\bar{L}_t = m^* I_t,$$

where the constant m^* is given by

$$m^* = \frac{1}{r_I} - \frac{(1-\gamma+\alpha_+\gamma)\delta}{\gamma K(\alpha_+ - 1)} \left(\frac{K\alpha_+\gamma}{(1-\gamma+\alpha_+\gamma)\hat{\beta}} \right)^{\frac{1}{1-\gamma}}.$$

I. Endogenous Default Boundary for the CRRA utility case

We *explicitly* characterize the endogenous default boundary (the endogenous credit limit):

- The endogenous credit limit denoted by $\bar{X}_t$ at time t can be written as the linear function of the *current* income level with some $m^* > 0$ *in equilibrium*:

$$X_t \geq \bar{X}_t := -m^* I_t, \tag{10}$$

  - The debt limit is changing over time.
  - The optimal wealth process never reaches the credit limit boundary thanks to the *liquidity* hedging component in the optimal portfolio:

$$\frac{\pi^*}{I_t} = \text{myopic demand} + \text{intertemporal hedging demand} + \text{liquidity demand}$$

    since the last term *drastically* decreases whenever the wealth-to-income ratio approaches to $-m^*$.

II. Endogenous Default Boundary - Income Growth

- Note $\frac{X_t}{I_t} \geq -m^*$, where

$$m^* = m^*(\mu_I, \sigma_I, \theta, \dots)$$

- If the mean return of income increases (higher μ_I) or the labor income has a lower risk (lower σ_I), the borrowing limit increases (higher m^*).

- Two Opposing Effects:
  - An increase in the expected income growth rate or an decrease in the income volatility relaxes the static budget constraint.
  - On the other hand, it tightens the endogenous credit constraint (or market participation constraint).

- The former outweigh the latter in the household investment decision while the opposite is true in the limited commitment literature.

Optimal Consumption Rate

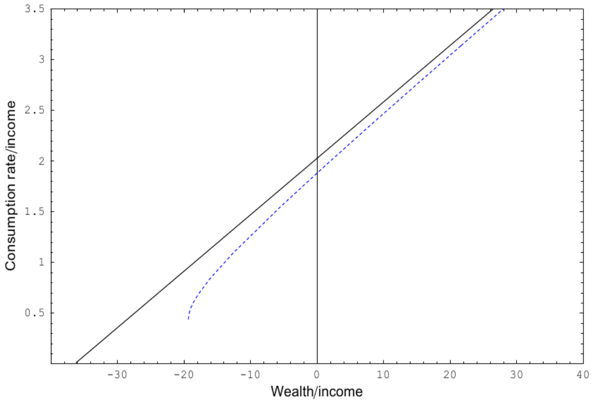


Figure: ($\gamma = 2,\ \beta = 0.07,\ r = 0.015,\ \mu = 0.08,\ \sigma = 0.15,\ \mu_I = 0.02,\ \sigma_I = 0.1$ and $\delta = 0.7$). *The dotted line shows the result of our model of the endogenous credit constraints and the solid line shows that with no constraint (The Merton solution).*

Optimal Risky Portfolio

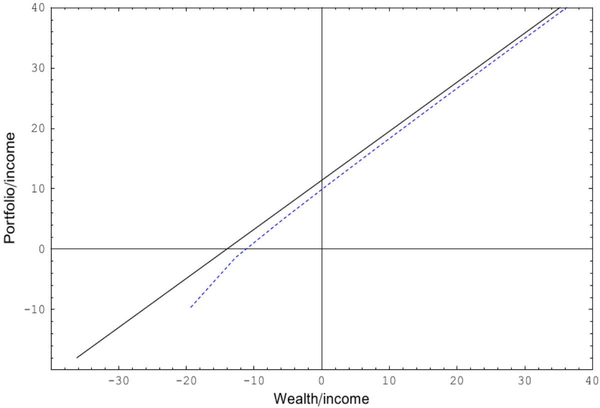


Figure: $(\gamma = 2,\ \beta = 0.07,\ r = 0.015,\ \mu = 0.08,\ \sigma = 0.15,\ \mu_I = 0.02,\ \sigma_I = 0.1$ and $\delta = 0.7)$. *The dotted line shows our model containing the endogenous credit constraint and the solid line shows the benchmark model.*

III. Business cycle & Credit limit

- The economy with worse investment opportunities tends to provide a greater debt limit for the individual investor.
- A decrease in the Sharpe ratio (θ) increases the borrowing limit since it discounts future income less.
- The rich tend to invest in risky assets more in downturns than they do in good times while the poor invest in risk assets more in good times than in bad times.
  1. One force: Risky investment gets higher as the Sharpe ratio becomes higher (Merton 1969, 1971).
  2. The other: The debt limit shrinks as the Sharpe ratio becomes higher.
  $$\theta \uparrow \quad \Longrightarrow \quad m^* \downarrow$$
  3. The reduction in the credit limit due to an increase in the Sharpe ratio rarely has impact on the investment decision of wealthy agents.
  4. On the other hand, it has a significant impact in the risky investment for the low wealth level due to the liquidity hedging term.

The optimal portfolio/income ratios for the CRRA utility function with different rate of returns of risky assets for different market conditions

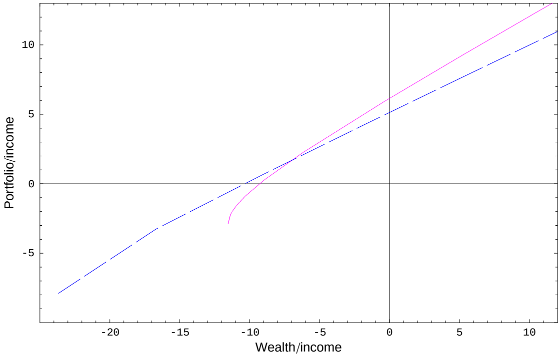


Figure: *The short-dotted line shows the model when the market price of risk is* $\theta = 0.28$ *(*$\mu = 0.1, r = 0.015, \sigma = 0.3$*) and the solid line shows the model with* $\theta = 0.45$ *(*$\mu = 0.2, r = 0.02, \sigma = 0.4$*). In this case, the endogenous minimum wealth-to-income ratios are given by* $m^* = -11.55$ *and* $m^* = 23.67$*, respectively.*

Unidversifiable Income Risk

- The income receives an unexpected negative shock

$$\epsilon_t = \begin{cases} \epsilon, & 0 \leq t \leq \tau, \\ k\epsilon, & \tau \leq t, \end{cases}$$

  where τ is the arrival time of un unexpected shock and k, $0 \leq k \leq 1$, is a recovery rate of income.
- The time τ is supposed to follow an exponential distribution with an intensity δ.
- The wealth dynamics evolves

$$dX_t = \begin{cases} (rX_t + \pi_t(\mu - r) - c_t + \epsilon_t)dt + \sigma\pi_t dB_t, & 0 \leq t \leq \tau, \\ (rX_t + \pi_t(\mu - r) - c_t + k\epsilon_t)dt + \sigma\pi_t dB_t, & \tau \leq t, \end{cases}$$

Unidversifiable Income Risk

- The autarky value in market participation constraint (3)

$$V(\bar{x}) = \mathbb{E}\left[\int_0^\infty e^{-\beta s} u((1-\phi)e_s)ds\right] = \frac{(1-\phi)^{1-\gamma}\epsilon^{1-\gamma}(1+\delta k^{1-\gamma})}{(1-\gamma)(\beta+\delta)}. \quad (11)$$

- From F.O.C., HJB equation becomes

$$-(\beta+\delta)V(x) + (rx - c + \epsilon)V'(x) - \frac{1}{2}\theta^2 \frac{V'(x)^2}{V''(x)} + \frac{\gamma}{1-\gamma}V'(x)^{-\frac{1-\gamma}{\gamma}} + \delta\frac{(x+k\epsilon/r)^{1-\gamma}}{K^\gamma(1-\gamma)} = 0,$$

with boundary condition (11).

Undiversifiable Income Risk

- Suppose

$$V'(x) = \lambda(x), \qquad G(\lambda(x)) = x + \frac{\epsilon}{r}.$$

Then

$$G'(\lambda(x))\lambda'(x) = 1,$$
$$G''(\lambda(x))\lambda'(x)^2 + G'(\lambda(x))\lambda''(x) = 0.$$

- By differentiating w.r.t. x and substituting into HJB equation, we have

$$-\frac{1}{2}\theta^2\lambda^2 G''(\lambda) - \lambda G'(\lambda)(\theta^2 + \beta + \delta - r) + rG(\lambda) + \frac{\delta}{K^\gamma}\left(G(\lambda) - \epsilon/r + k\epsilon/r\right)^{-\gamma} G'(\lambda) = \lambda^{-\frac{1}{\gamma}}. \quad (12)$$

Undiversifiable Income Risk

- Conjecture the solution to (12) as

$$G(\lambda) = \frac{\gamma\lambda^{-\frac{1}{\gamma}}}{\gamma A + \delta} + \alpha(\lambda)\lambda^{-n_+} + \beta(\lambda)\lambda^{-n_-},$$

subject to

$$\alpha'(\lambda)\lambda^{-n_+} + \beta'(\lambda)\lambda^{-n_-} = 0,$$

where n_+ and n_- are two real roots of the quadratic equation

$$-\frac{1}{2}\theta^2 n(n+1) + (\beta + \delta - r)n + r = 0.$$

- If we substitute the conjectured form into the nonlinear ODE, we have

$$\frac{1}{2}\theta^2\left(n_+\alpha'(\lambda)\lambda^{-n_+ +1} + n_-\beta'(\lambda)\lambda^{-n_- +1}\right) + \frac{\delta}{K^\gamma}\left(G(\lambda) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{-\gamma} G'(\lambda) = 0. \quad (13)$$

Undiversifiable Income Risk

- The coefficients of the conjectured form are derived by

$$\alpha(\lambda) = -\frac{2\delta}{K^{\gamma}\theta^2(n_+ - n_-)}\int_0^{\lambda} \mu^{n_+ - 1}\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{-\gamma} G'(\mu)d\mu,$$

$$\beta(\lambda) = \beta(\bar{\lambda}) - \frac{2\delta}{K^{\gamma}\theta^2(n_+ - n_-)}\int_{\lambda}^{\bar{\lambda}} \mu^{n_- - 1}\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{-\gamma} G'(\mu)d\mu,$$

- By integration by parts, the solution to the nonlinear ODE (13) is given by

$$\begin{aligned} G(\lambda) = {} & \frac{\gamma\lambda^{-\frac{1}{\gamma}}}{\gamma K + \delta} + \left(\beta(\bar{\lambda}) - \frac{2\delta\bar{\lambda}^{n_- - 1}}{K^{\gamma}\theta^2(n_+ - n_-)}\frac{\left(G(\bar{\lambda}) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}\right)\lambda^{-n_-} \\ & + \frac{2\delta\lambda^{n_- - 1}}{K^{\gamma}\theta^2(n_+ - n_-)}\left((n_+ - 1)\lambda^{-n_+}\int_0^{\lambda}\mu^{-n_+ - 2}\frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}d\mu\right. \\ & \left. + (n_- - 1)\lambda^{-n_-}\int_{\lambda}^{\bar{\lambda}}\mu^{-n_- - 2}\frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}d\mu\right). \end{aligned} \tag{14}$$

Undiversifiable Income Risk

- The free boundary $\bar{\lambda}$ is obtained from the following boundary conditions

  1. The participation constraint

  $$V(\bar{x}) = H(\bar{\lambda}) = \frac{(1-\phi)^{1-\gamma}\epsilon^{1-\gamma}(1+\delta k^{1-\gamma})}{(1-\gamma)(\beta+\delta)},$$

  where $H(\lambda)$ is defined by

  $$H(\lambda) \triangleq \frac{1}{\beta+\delta}\left\{ rG(\lambda)\lambda - \frac{1}{2}\theta^2\lambda^2 G'(\lambda) + \frac{\gamma}{1-\gamma}\lambda^{-\frac{1-\gamma}{\gamma}} + \frac{\delta\left(G(\lambda) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{K^{\gamma}(1-\gamma)} \right\}$$

  2. No portfolio at the minimum wealth level

  $$G'(\bar{\lambda}) = 0$$

Undiversifiable Income Risk

- The boundary conditions are transformed to
  - The participation constraint

$$\frac{(1-\phi)^{1-\gamma}\epsilon^{1-\gamma}(1+\delta K^{1-\gamma})}{(1-\gamma)r\bar{\lambda}} - \frac{\gamma}{(1-\gamma)r}\bar{\lambda}^{-\frac{1}{\gamma}} - \frac{\delta\left(G(\bar{\lambda})-\frac{\epsilon}{r}-\frac{k\epsilon}{r}\right)^{1-\gamma}}{K^{\gamma}(1-\gamma)\bar{\lambda}}\left(\frac{1}{r}-\frac{2n_+}{\theta^2 n_-(n_+-n_-)}\right)$$
$$= \frac{\bar{\lambda}^{-\frac{1}{\gamma}}}{\gamma K+\delta}\left(\gamma-\frac{\bar{\lambda}}{n_+}\right) + \frac{2\delta(n_+-1)\bar{\lambda}^{-n_+}}{K^{\gamma}\theta^2 n_-}\int_0^{\bar{\lambda}}\mu^{n_+-2}\frac{\left(G(\mu)-\frac{\epsilon}{r}+\frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}d\mu \tag{15}$$

  - No investment at the boundary

$$0 = -\frac{\bar{\lambda}^{-\frac{1}{\bar{\lambda}}-1}}{\gamma K+\delta} - n_-\beta(\bar{\lambda})\bar{\lambda}^{-n_--1} - \frac{2\delta\bar{\lambda}^{-2}\left(G(\bar{\lambda})-\frac{\epsilon}{r}+\frac{k\epsilon}{r}\right)^{1-\gamma}}{K^{\gamma}\theta^2(1-\gamma)}\left(\frac{1-n_+-n_-}{n_+-n_-}\right)$$
$$-\frac{2\delta n_+(n_+-1)\bar{\lambda}^{-n_+-1}}{K^{\gamma}\theta^2(n_+-n_-)}\int_0^{\bar{\lambda}}\mu^{n_+-2}\left(G(\mu)-\frac{\epsilon}{r}+\frac{k\epsilon}{r}\right)^{1-\gamma}d\mu. \tag{16}$$

Undiversifiable Income Risk

To get the free boundary numerically, we apply the iterative method

1. Find the values of $\beta(\bar{\lambda})$ from (16) and substitute into $G(\lambda)$ in (14) by imposing $\delta = 0$. Then we can get an initial free-boundary value $\bar{\lambda}$ from the equation (15).
2. Obtain $\beta(\bar{\lambda})$ from the boundary condition (16) by using initial $G(\lambda)$ and $\bar{\lambda}$.
3. Update $G(\lambda)$ in (14) and find new $\bar{\lambda}$ from the equation (15).
4. Repeat until $\bar{\lambda}$ converges.

Undiversifiable Income Risk

- Let the variable λ^* be the solution to

$$x + \frac{\epsilon}{r} = \frac{\gamma {\lambda^*}^{-\frac{1}{\gamma}}}{\gamma K + \delta} + B(\bar{\lambda}){\lambda^*}^{-n_-}$$
$$+ \frac{2\delta}{K^\gamma \theta^2 (n_+ - n_-)} \left((n_+ - 1){\lambda^*}^{-n_+} \int_0^{\lambda^*} \mu^{n_+ - 2} \frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma} d\mu \right.$$
$$\left. + (n_- - 1){\lambda^*}^{-n_-} \int_{\lambda^*}^{\bar{\lambda}} \mu^{n_- - 2} \frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma} d\mu \right),$$

where

$$B(\bar{\lambda}) = \beta(\bar{\lambda}) - \frac{2\delta \bar{\lambda}^{n_- - 1}}{K^\gamma \theta^2 (n_+ - n_-)} \frac{\left(G(\bar{\lambda}) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}.$$

Undiversifiable Income Risk

- The optimal consumption rate

$$c_t^* = \left(K + \frac{\delta}{\gamma}\right)\left(x + \frac{\epsilon}{r}\right) - B(\bar{\lambda})\left(A + \frac{\delta}{\gamma}\right)\lambda^{*\,-n_-}$$
$$+ \frac{2(K+\delta/\gamma)\delta}{K^\gamma \theta^2 (n_+ - n_-)}\left((n_+ - 1)\lambda^{*\,-n_+}\int_0^{\lambda^*}\mu^{n_+-2}\frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}d\mu\right.$$
$$\left.+ (n_- - 1)\lambda^{*\,-n_-}\int_{\lambda^*}^{\bar{\lambda}}\mu^{n_--2}\frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}d\mu\right).$$

- The optimal investment in the risky asset

$$\pi_t^* = \frac{\theta}{\gamma\sigma}\left(x + \frac{\epsilon}{r}\right) + \frac{\theta}{\sigma}\left(n_- - \frac{1}{\gamma}\right)\lambda^{*\,-n_-} - \frac{2\delta}{\sigma\theta K^\gamma(1-\gamma)}\frac{1}{\lambda^*}\left(x + \frac{k\epsilon}{r}\right)^{1-\gamma}$$
$$+ \frac{2\delta}{\sigma\theta K^\gamma (n_+ - n_-)}\left(\left(n_+ - \frac{1}{\gamma}\right)(n_+ - 1)\lambda^{*\,-n_+}\int_0^{\lambda^*}\mu^{n_+-2}\frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}d\mu\right.$$
$$\left.+ \left(n_- - \frac{1}{\gamma}\right)(n_- - 1)\lambda^{*\,-n_-}\int_{\lambda^*}^{\bar{\lambda}}\mu^{n_--2}\frac{\left(G(\mu) - \frac{\epsilon}{r} + \frac{k\epsilon}{r}\right)^{1-\gamma}}{1-\gamma}d\mu\right).$$

Conclusion

This paper presents a continuous-time consumption-investment model for a household without debt enforcement. Using a duality approach, we derive closed-form solutions for optimal consumption and portfolio policies and for endogenous credit (debt) limits over time.

Moreover, we consider the incomplete market model with undiversifiable income risk. The dynamic programming principle is applied to obtain the optimal policies and the endogenous credit limit with semi-explicit form.

The empirical implications based on the theoretical results:

(1) The household's **current level of income** is particularly important for determining **the unsecured borrowing limit**
(2) The unsecured borrowing limit is **greater** for a household with **a higher income growth rate and lower income volatility**.
(3) The unsecured borrowing limit is **greater in a boom** than in a recession.
(4) The portfolio **share of risky assets for the rich** and the poor exhibit **different patterns across booms and recessions.**