Fair valuation in life insurance

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Outline

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- ▷ Mortality risk
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Role of actuary – kinds of actuaries

- Bühlmann made a distinction between three kinds of actuaries:
  - ▷ First kind: all processes are deterministic (as in current life insurance actuarial practice, with fixed valuation interest rate and fixed mortality table)
  - ▷ Second kind: the insurance claim processes are stochastic (as in current Property&Casualty actuarial practice, with fixed valuation interest rate and stochastic processes for claims)
  - ▷ Third kind: both insurance processes and financial components are stochastic (we want to move to that direction)

Reason interest in fair value

- IASB is currently (since 2004) aiming for use of fair valuation in insurance industry
- New solvency regulations (EU: Solvency II) are moving to fair value techniques (France, UK, Netherlands, Denmark, Switzerland, Canada)
- Fair value is important technique for risk management
- Modern financial economics uses similar techniques for valuation complex financial instruments
- Some complex products can only be understood using these techniques

Definition (IASB): Fair value

"The amount for which an asset could be exchanged or a liability settled between knowledgeable, willing parties in an arm's length transaction"

– Definition is still rather vague to start actual calculations

IASB: fair valuation

- Assets and liabilities at "Market value" or "Fair value".

1. If you have reliable market prices use these.
   - ▷ Reliable: liquid markets
2. If you have no market prices, but are able to replicate, then the price of the replicating portfolio is the price of the instrument.
   - ▷ Replication: cash flow matching
3. If neither 1 or 2 applies, make a model to price the instrument as consistently as possible. Use of a "Market Value Margin" is necessary.

Fair valuation: (in)complete market

- We assume financial markets are complete: you can trade everything at the required volume without influencing the price
- Non financial markets (mortality risks, PC risks) are incomplete: there is no active market for these risks
- In incomplete markets all agents price a product using their own utility function
- In incomplete markets a MVM is required to reflect the non hedgeable risk in a product

History of the equity-linked life insurance contracts

- Appears to have been introduced first in the Netherlands in 1953.
- The first equity-based product in the United Kingdom was sold in 1957.
- In Canada, equity-linked policies have been issued since 1967.
- In the United States, where they are known as variable life insurance policies, the equity-linked life insurance business started at the end of 1970s.
- In Germany, the first equity-linked insurance policy was sold in August 1996 by a foreign provider.

Equity-linked life insurance

- ... combines life insurance with an investment strategy

> ▷ maintains the properties of an insurance contract
>
> ▷ accumulates capital by means of an investment portfolio.
>
> ◇ The benefits of the contract are strongly linked to performance of the capital market.
>
> ◇ In contrast to the pure investment in the capital market, the contract holder only participates in the upwards-movements of the capital market.

Surplus participation

- The benefit of this contract is determined by

▷ a guaranteed amount

▷ a participation in the surpluses of the portfolio (bonus).

◇ The surplus participation gives a redistribution of the achieved surpluses between the insured and the insurance company

∘ the redistribution rule of the surpluses is stipulated in the contract.

◇ The part of the surpluses that the insured gives up can be considered as implicit premiums

⇒ Possible that the explicit premium payments of the insured is limited in its size

Risks involved

- Life insurance contracts are usually long-term contracts
- The payoff time depend on the residual life time of the insured
- Equity-linked $\Rightarrow$ the insurance benefits are determined ex post, because it depends on the realization of the stock market

$\Rightarrow$ Analysis of life insurance shall take account of

- mortality/longevity risk
- Financial market risk related to the asset
- Interest rate risk

Equity-linked investment with a minimum guarantee

- Description 1

$$\begin{aligned}
&\max\left\{\frac{A}{S(t_0)}S(T), A\cdot\exp\{gT\}\right\}\\
=\ & A\cdot\exp\{g\cdot T\}+\max\left\{A\cdot\frac{S(T)}{S(t_0)}-A\cdot\exp\{g\cdot T\},0\right\}\\
=\ & A\cdot\exp\{g\cdot T\}+A\cdot\max\left\{\frac{S(T)}{S(t_0)}-\exp\{g\cdot T\},0\right\}.
\end{aligned}$$

- Interpretation
  - Investment with a fixed guaranteed interest rate g
  - plus a bonus (proportional to the initial investment), when the rate of return of the portfolio exceeds the interest rate guarantee $g \Rightarrow$ European call option

Equity-linked investment with a minimum guarantee

- Description 2

$$\max\left\{\frac{A}{S(t_0)}S(T), A\cdot\exp\{g\cdot T\}\right\}$$
$$= \quad \frac{A}{S(t_0)}\cdot S(T) + A\max\left\{\exp\{g\cdot T\} - \frac{S(T)}{S(t_0)}, 0\right\}.$$

- Interpretation
  - Investment in a risky portfolio,
  - and additionally a payment which is proportional to the initial investment, when the rate of the return of the risky asset is below the guaranteed interest rate $g \Rightarrow$ European put option

Fair contract?

- $g = r$ leads to no fair contract!

$$
\begin{aligned}
A &= A \cdot \exp\{(g-r)\cdot T\} + A \cdot E^* \left[e^{-rT} \max\left\{ \frac{S(T)}{S(t_0)} - \exp\{g \cdot T\}, 0 \right\} \right] \\
&> A \cdot \exp\{(g-r)\cdot T\}
\end{aligned}
$$

$\Rightarrow$ $g < r$.

Fair contract?

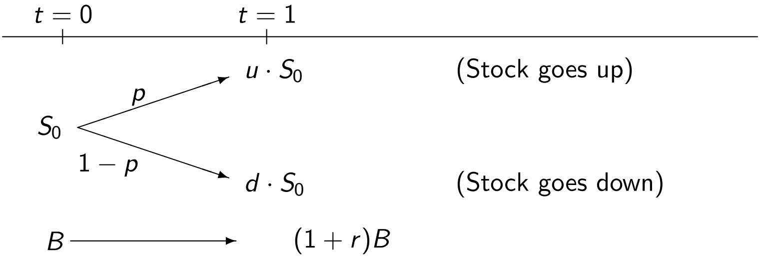
- When the asset is fair, the resulting contract cannot be fair! Based on the second description

$$
\begin{aligned}
&\frac{A}{S(t_0)} \cdot E^*[e^{-rT} S(T)] + A \cdot E^* \left[e^{-rT} \max \left\{ \exp\{g \cdot T\} - \frac{S(T)}{S(t_0)}, 0 \right\} \right] \\
=&\frac{A}{S(t_0)} \cdot S(t_0) + A \cdot E^* \left[e^{-rT} \max \left\{ \exp\{g \cdot T\} - \frac{S(T)}{S(t_0)}, 0 \right\} \right] \\
>&A
\end{aligned}
$$

$\Rightarrow$ The customer has to pay more than the initial amount to obtain the above guarantee.

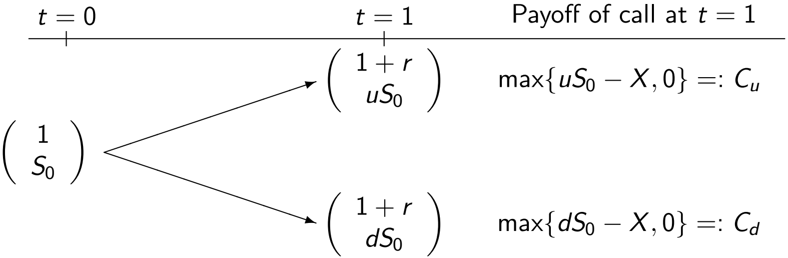
Arbitrage-free pricing, risk-neutral pricing

- Arbitrage: make a profit without a risk
- Basic idea of option pricing: no arbitrage + replication
  - ◇ Find a strategy that replicates the payoff of a call option ⇒ the price of the replicating portfolio = price of call (Law of One Price)
- 1-period binomial tree: stock S and Bond B ($d < 1 + r < u$)



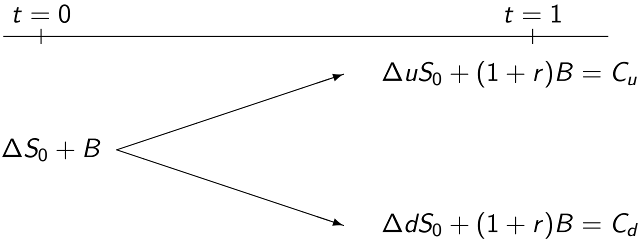
Payoff of the European call option

The buyer of a European call option has the right, but not the obligation to buy an agreed quantity of a particular commodity or financial instrument (the underlying S) from the seller of the option at a certain time (the expiration date) for a certain price (the strike price X)



Replication

- Find a strategy whose value at time 1 replicates that of the call ⇒ No arbitrage implies that the value of the call shall correspond to that of the strategy at time 0
- Δ the number of the stock and B amount in bond



$$\Delta = \frac{C_u - C_d}{(u-d)S_0}$$

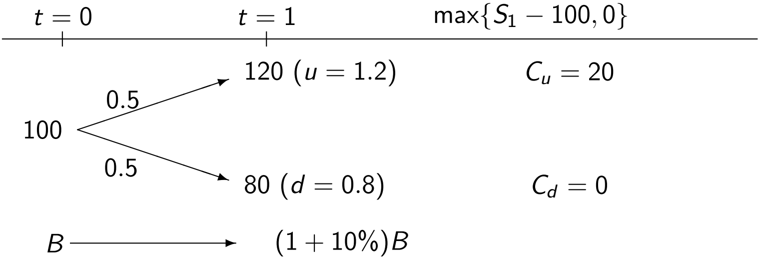
$$B = \frac{1}{1+r}\left(C_u - \Delta u S_0\right) = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d}.$$

Price of call option

- Under the no-arbitrage condition, the initial price of the call equals the initial value of the replicating strategy:

$$
\begin{aligned}
C_0 =& \text{Strategy value} \\
=& \Delta S_0 + B \\
=& \frac{C_u - C_d}{u - d} + \frac{1}{1+r} \frac{uC_d - dC_u}{u - d}
\end{aligned}
$$

Example



- Replication of the call through:

$$\Delta \cdot 120 + \eta(1 + 10\%) = 20; \quad \Delta \cdot 80 + \eta(1 + 10\%) = 0;$$

$$\Rightarrow \Delta = 0.5, \; B = -36.3636$$

$$\Rightarrow C_0 = 0.5 \cdot 100 - 36.3636 = 13.6364$$

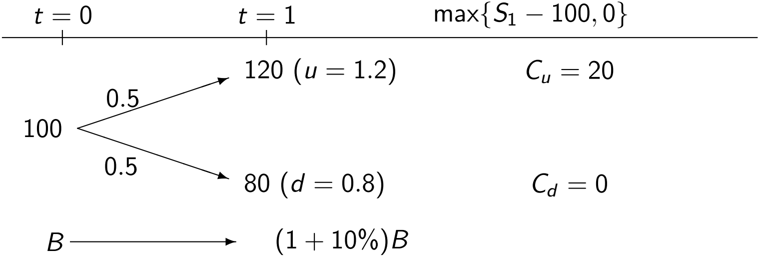
Price of call option

- Under the no-arbitrage condition, the initial price of the call equals the initial value of the replicating strategy:

$$
\begin{aligned}
C_0 = \text{Strategy value} =& \frac{C_u - C_d}{u-d} + \frac{1}{1+r}\frac{uC_d - dC_u}{u-d} \\
=& \frac{1}{1+r}\left(\frac{1+r-d}{u-d}C_u + \frac{u-(1+r)}{u-d}C_d\right) \\
=& \frac{1}{1+r}\left(p^* C_u + (1-p^*)C_d\right)
\end{aligned}
$$

  with $p^* = \frac{1+r-d}{u-d}$. $p^* \in]0,1[$ if $u > 1+r > d$ and owns "probability interpretation".

- The probability measure p^* is often called "equivalent martingale measure" or "risk-neutral measure"



- Alternative: Risk-neutral valuation

$$p^* = \frac{1 + r - d}{u - d} = 3/4, \quad 1 - p^* = \frac{1}{4}$$

$$\Rightarrow C_0 = \tfrac{1}{1+r}\left(\tfrac{3}{4} \cdot 20 + \tfrac{1}{4} \cdot 0\right) = 13.6364$$

Comments

- The valuation of the options can follow two approaches:
  - The first is based on replicating argument
  - The second determines the price as expected discounted value of the terminal payoff under the risk neutral probability measure P^*.
    - The determination depends only on P^* and does not need to determine the strategy
- Real world measure P does not appear in the formula
- No arbitrage is equivalent to the existence of a probability measure P^*
- Under the risk-neutral measure, we have

$$E^*[S_1] = S_0(1+r)$$

Assumptions of the Black and Scholes Model

- Goal is to price a European call option with a payoff $\max\{S_T - K, 0\}$
- Assumptions

  - ▷ The asset pays no dividends during the option's life
  - ▷ European exercise terms are used
  - ▷ Markets are efficient
  - ▷ No commissions are charged
  - ▷ Interest rates remain constant and known
  - ▷ The underlying asset is lognormally distributed

- The stochastic process $\{W(t)\}_{t\in[0,T]}$ is a standard Brownian motion, when it holds

  - $W(0) = 0$
  - The increments (W(t)-W(s)) are normally distributed with mean zero and variance $(t-s)$, i.e.,

  $$W(t) - W(s) \sim N(0, t-s) \quad \forall t > s$$

  - For $s < t \leq u < v$, the increments $(W(t) - W(s))$ and $(W(v) - W(u))$ are stochastically independent.
  - Almost surely, $t \longmapsto W(t)$ is continuous on $t \in [0, T]$

No arbitrage and existence of equivalent martingale measure

- In a complete financial market, e.g. in a Black and Scholes Economy, no arbitrage condition is equivalent to the existence of a unique equivalent martingale measure P^* (risk-neutral probability measure given a certain numeraire).
- No arbitrage condition has a consequence that the instantaneous rate of return of all the assets shall correspond to the risk free interest rate.

- Under the risk-neutral probability measure P^*, the asset $(S(t))_{t\in[0,T]}$ is assumed to evolve according to the following stochastic differential equation:

$$dS(t) = rS(t)dt + \sigma S(t)dW^*(t)$$

  - ▷ $(W^*(t))_{t\in[0,T]}$ is a Browian motion under P^*
  - ▷ $\sigma > 0$ is the volatility of the asset

$$\sigma^2 := \lim_{dt\to 0} \frac{1}{dt} \mathrm{Var}^* \left[\frac{dS(t)}{S(t)} \middle| S(t) \right]$$

Some results

- The solution to the SDE is given by

$$S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\cdot t + \sigma\cdot(W^*(t) - W^*(0))\right\}$$

- Under the risk-neutral probability measure P^*, it holds

$$\begin{aligned} E^*[S(T)] &= S(0)\exp\{rT\} \\ \mathrm{Var}^*[S(T)] &= (S(0))^2\exp\{2\,r\,T\}(\exp\{\sigma^2\,T\} - 1) \end{aligned}$$

No arbitrage price

- The arbitrage-free price of a T-contingent claim $X(T)$ at time t $(t \leq T)$ corresponds to the expected discounted final payment under the risk-neutral probability measure P^* conditional on $\mathcal{F}_t$, i.e.

$$X_0 = E^*\left[e^{-rT}X(T)\right]$$

where E^* is the expectation taken under P^*

- The price of the European with a payoff $\max\{S_T - K, 0\}$ is given by

$$
\begin{aligned}
C(S(0), K, 0, T) &= E^*\left[e^{-rT}\max\{S(T)-K,0\}\right] \\
&= S(0)N(d_1) - Ke^{-rT}N(d_2) \\
d_{1/2} &= \frac{\ln\frac{S(0)}{Ke^{-rT}} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}
\end{aligned}
$$

▷ where $N(t) = \int_0^t \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

Proportional maturity participation

- The benefit of the contract at maturity T is determined by

$$
\begin{aligned}
&(1-\alpha)\cdot A\cdot\exp\{g\cdot T\}+\alpha\cdot\max\left\{\frac{A}{S(t_0)}\cdot S(T),A\cdot\exp\{g\cdot T\}\right\}\\
=&A\exp\{gT\}+\alpha\cdot A\cdot\max\left\{\frac{A}{S(t_0)}\cdot S(T)-A\cdot\exp\{g\cdot T\},0\right\}
\end{aligned}
$$

  - ▷ $\alpha=1 \Rightarrow$ No fair contract.
  - ▷ for $g=r$, α must be zero in order to achieve a fair contract.

Fair contract parameter

- Condition for a fair combination of g and α

$$
\begin{aligned}
A &= A \cdot \exp\{(g-r)\cdot T\} \\
&\quad + \alpha \cdot A \cdot E^{*}\left[e^{-rT} \max\left\{ \frac{S(T)}{S(t_0)} - \exp\{g \cdot T\}, 0 \right\} \right] \\
\Leftrightarrow \alpha(g) &= \frac{1-\exp\{(g-r)\cdot T\}}{E^{*}\left[e^{-rT} \max\left\{ \frac{S(T)}{S(t_0)} - \exp\{g \cdot T\}, 0 \right\} \right]} \\
\text{and } \alpha(r) &= 0.
\end{aligned}
$$

Result I

- Under the assumptions of Black and Scholes model, the present value of the maturity participation is determined by

$$
\begin{aligned}
& E^*\left[e^{-rT}\max\left\{\frac{S(T)}{S(0)}-\exp\{g\cdot T\},0\right\}\right] \\
=& N(d_1)-\exp\left\{(g-r)\cdot T\right\}\cdot N(d_2), \\
d_{1/2} =& \frac{(r-g\pm 1/2\cdot\sigma^2)\cdot T}{\sigma\sqrt{T}}.
\end{aligned}
$$

Result II

- Under this model assumptions, the set of fair contracts with a minimum interest rate guarantee $g \leq r$ and a proportional maturity participation α is characterized by

$(g, \alpha^*(g))$ with

$$\alpha^*(g) = \frac{1 - \exp\{(g-r)\cdot T\}}{N(d_1) - \exp\{(g-r)\cdot T\}\cdot N(d_2)}$$

$$d_{1/2} = \frac{(r - g \pm 1/2\cdot\sigma^2)\cdot T}{\sigma\sqrt{T}}$$

Example

Fair equity-linked contract with a single premium with proportional maturity participation volatility 25%, market interest rate 10%								
Guaranteed interest rate	Proportional maturity participation $\alpha^*(g)$							
g	Duration in years							
	1	2	3	4	5	10	15	20
-0.01	0.67068	0.79438	0.85786	0.89658	0.92232	0.97644	0.99143	0.99662
0.00	0.63543	0.76207	0.82958	0.87206	0.90111	0.96596	0.98611	0.99389
0.01	0.59648	0.72485	0.79592	0.84204	0.87444	0.95121	0.9778	0.98916
0.02	0.55346	0.682	0.75587	0.8053	0.84096	0.93055	0.96497	0.98113
0.03	0.50593	0.63265	0.70824	0.76036	0.79897	0.90175	0.94536	0.96774
0.04	0.45343	0.57582	0.6516	0.70544	0.74637	0.86178	0.91565	0.94571
0.05	0.39542	0.51038	0.58424	0.6383	0.68049	0.80645	0.87093	0.90989
0.06	0.33132	0.43501	0.50414	0.55627	0.59800	0.73001	0.80394	0.85216
0.07	0.26049	0.34819	0.40887	0.456	0.49471	0.62449	0.7039	0.75968
0.08	0.18221	0.24817	0.29552	0.33341	0.36535	0.47888	0.55472	0.61201
0.09	0.09567	0.13289	0.16062	0.18348	0.20326	0.27785	0.33225	0.37642
0.10	0	0	0	0	0	0	0	0
0.11	-0.10579	-0.15326	-0.19136	-0.22471	-0.25511	-0.38476	-0.49756	-0.60364

Result

- The participation rate α is monotonically decreasing in the guaranteed interest rate g.
- The participation rate α is monotonically increasing in the maturity T for $g < r$.
- When the guaranteed interest rate coincides with the market interest rate, i.e., $g = r$, the fair participation rate α shall be equal to zero.
- If the guaranteed interest rate exceeds the market interest rate, i.e., $g > r$, a negative fair participation rate results.

Critique on the proportional maturity participation

- In case of a maturity participation, the bonus part
  - ▷ depends on the asset value at the maturity date T only
  - ▷ the payment which exceeds the guaranteed interest rate g results when the long-term rate of return $\frac{1}{T} \ln \left(\frac{S(T)}{S(0)} \right)$ lies above g.
  - ▷ a sudden fall of the asset value at the maturity date T might lead to a substantial loss (also when the asset values are very high shortly before the maturity date).

Direct surplus participation

- Necessary to link the surplus participation to the asset evolution during $[0, T]$.

  - The surplus participation shall be determined each period.
  - The surplus participation will be connected to the portfolio strategy.

- Assumptions:

  - Proportional direct surplus participation at time point t_{i+1}, $i = 0, \cdots, M-1$, with $t_M = T$.
  - In a contract with a single premium A (paid at the time point 0), usually the surpluses are not redistributed to the investor at time t_{i+1}.
  - Conventionally all the surpluses are provided to the insured at the maturity date.

...Direct surplus participation

- Assumption : Each single surplus at time t_{i+1}, $i = 0, \cdots, M-1$ is accumulated to the maturity date with the interest rate g.

$\Rightarrow$ The benefits of the contract at maturity T is given by

$$A \cdot e^{g \cdot T} + \alpha \cdot A \sum_{i=0}^{M-1} e^{g \cdot (T - t_{i+1})} \cdot \frac{S(t_i)}{S(0)} \max\left\{ \frac{S(t_{i+1})}{S(t_i)} - e^{g \cdot \Delta t}, 0 \right\}$$

Questions

1) What is the difference between direct and maturity surplus participation?

2) How can the set of fair contracts be characterized, i.e. what combinations of (g, α) lead to a fair contract?

Reply 1)

$$\alpha \cdot A \sum_{i=0}^{M-1} e^{g\cdot(T-t_{i+1})} \frac{S(t_i)}{S(0)} \cdot \max\left\{ \frac{S(t_{i+1})}{S(t_i)} - e^{g\cdot\Delta t}, 0\right\}$$

$$\begin{aligned}
&= \alpha \cdot \frac{A}{S(0)} \sum_{i=0}^{M-1} \max\left\{ S(t_{i+1}) \cdot e^{g\cdot(T-t_{i+1})} - S(t_i) \cdot e^{g\cdot(T-t_i)}, 0\right\} \\
&\geq \alpha \cdot \frac{A}{S(0)} \max\left\{ \sum_{i=0}^{M-1} \left(S(t_{i+1}) \cdot e^{g\cdot(T-t_{i+1})} - S(t_i) \cdot e^{g\cdot(T-t_i)}\right), 0\right\} \\
&= \alpha \cdot \frac{A}{S(0)} \max\left\{ \sum_{i=0}^{M-1} S(t_{i+1}) \cdot e^{g\cdot(T-t_{i+1})} - \sum_{i=0}^{M-1} S(t_i) \cdot e^{g\cdot(T-t_i)}, 0\right\} \\
&= \alpha \cdot \frac{A}{S(0)} \max\left\{ \sum_{i=1}^{M} S(t_i) \cdot e^{g\cdot(T-t_i)} - \sum_{i=0}^{M-1} S(t_i) \cdot e^{g\cdot(T-t_i)}, 0\right\} \\
&= \alpha \cdot \frac{A}{S(0)} \max\left\{ S(t_M) - S(0) \cdot e^{g\cdot T}, 0\right\}
\end{aligned}$$

Summary

- A proportional direct surplus participation accumulated with the interest rate guarantee g leads to payment which never lies under a maturity surplus participation (all the other parameters or conditions are identical).

◇ If (g_s^*, α_s^*) leads to a fair contract under proportional maturity participation

◇ and (g_d^*, α_d^*) a fair combination under proportional direct surplus participation

$\Rightarrow$ so for $g_d^* = g_s^*$ it holds necessarily : $\alpha_d^* \leq \alpha_s^*$.

Reply 2): Fair combination (g_d^*, α_d^*)

- $\cdots$ of a proportional direct surplus participation:

$$\begin{aligned}\alpha_d^* &= \alpha(g_d^*) = \frac{1-\exp\{(g_d^*-r)\cdot T\}}{R_d(g_d^*)} \quad \forall g_d^* \in]-\infty, r] \\ R_d(g_d^*) &:= E^*\left[e^{-rT}\sum_{i=0}^{M-1} e^{g\cdot(T-t_{i+1})}\frac{S(t_i)}{S(0)}\left(\frac{S(t_{i+1})}{S(t_i)} - e^{g\cdot\Delta t}, 0\right)^+\right]\end{aligned}$$

Result

- The set of fair contracts with proportional direct surplus participation α and minimum interest rate guarantee g is determined by

$$\alpha^*(g_d) = \frac{1 - e^{(g_d - r)\Delta t}}{N(d_1) - e^{(g_d - r)\Delta t} N(d_2)}$$

$$\text{mit} \quad d_{1/2} := \frac{(r - g_d \pm \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}$$

$\diamond$ The fair participation rate is independent of T. It only depends on the length of the time interval $\Delta t = t_{i+1} - t_i$

Example

Fair contracts with a single premium and a proportional direct surplus participation Volatility 25%, Market interest rate 10%, Duration 5 years						
Minimum interest rate guarantee	Proportional direct surplus participation $\alpha_d^*(g)$					
g	Number of participation each year					
	12	6	4	3	1	Single premium Maturity participation $\alpha_s^*(g)$
-0.01	0.27289	0.36296	0.42453	0.47187	0.67068	0.92232
0.00	0.2515	0.33627	0.39483	0.44023	0.63543	0.90111
0.01	0.22949	0.30847	0.36361	0.40672	0.59648	0.87444
0.02	0.20684	0.27952	0.3308	0.37124	0.55346	0.84096
0.03	0.18352	0.24936	0.29631	0.33364	0.50593	0.79897
0.04	0.15952	0.21795	0.26006	0.29382	0.45343	0.74637
0.05	0.13482	0.18523	0.22195	0.25164	0.39542	0.68049
0.06	0.10939	0.15115	0.18189	0.20695	0.33132	0.598
0.07	0.08322	0.11565	0.13977	0.15961	0.26049	0.49471
0.08	0.05628	0.07867	0.0955	0.10946	0.18221	0.36535
0.09	0.02855	0.04014	0.04894	0.05631	0.09567	0.20326
0.10	0	0	0	0	0	0
0.11	-0.02938	-0.04182	-0.05146	-0.05967	-0.10579	-0.25511

Observations

- The participation rate is monotonically decreasing in the interest rate guarantee.
- The participation rate is monotonically decreasing in the number of participation each year.
- If it is assumed that the interest rate guarantee lies below the market interest rate, it implies that the participation rate in a direct surplus participation is smaller than that in a maturity participation (all the other parameters are identical).

Observations

- $g = r$ leads to a fair participation rate α of zero.
- $g > r$ leads to a negative participation rate α which is smaller than that in case of surplus participation.
- In case of a contract with a direct surplus participation, the participation rate is independent of the duration of the contract T.

Constrained surplus participation I

- In addition to the proportional surplus participation which potentially can result in an unconstrained surplus participation, the insurance company also provides contracts with a constrained surplus participation.
- This can be named as constrained surplus participation.
- Constrained direct surplus participation at time t_{i+1} with $u > g$:

$$\min\left\{\max\left\{\frac{A}{S(0)}\left(S(t_{i+1}) - S(t_i)\cdot e^{g\cdot\Delta t}\right), 0\right\}, \frac{A}{S(0)}S(t_i)(e^{u\cdot\Delta t} - e^{g\cdot\Delta t})\right\}$$

$$= \quad A\cdot\frac{S(t_i)}{S(0)}\left(\max\left\{\frac{S(t_{i+1})}{S(t_i)} - e^{g\cdot\Delta t}, 0\right\} - \max\left\{\frac{S(t_{i+1})}{S(t_i)} - e^{u\Delta t}, 0\right\}\right)$$

Constrained surplus participation II

- Assumption: again the periodic surpluses at t_{i+1}, $i = 0, \cdots, M-1$ are accumulated to the maturity date T with the interest rate g.
- The total benefit of the contract at the maturity date is given by

$$Ae^{g\cdot T} + \sum_{i=0}^{M-1} e^{g\cdot(T-t_{i+1})} A \frac{S(t_i)}{S(0)}$$
$$\cdot \left(\max\left\{ \frac{S(t_{i+1})}{S(t_i)} - e^{g\cdot\Delta t}, 0 \right\} - \max\left\{ \frac{S(t_{i+1})}{S(t_i)} - e^{u\cdot\Delta t}, 0 \right\} \right)$$

Fair combination of (g^*, u^*) (I)

- The set of the fair combinations (g^*, u^*) in case of a constrained surplus participation can be expressed as the solution of the contract:

$$1 = e^{(g-r)T} + \sum_{i=0}^{M-1} e^{(g-r)(T-t_{i+1})} \left(N(d_1) - e^{(g-r)\Delta t} N(d_2) - N(\tilde{d}_1) + e^{(u-r)\Delta t} N(\tilde{d}_2) \right)$$

Fair combination (g^*, u^*) (II)

- Further, it holds

$$
\begin{aligned}
\frac{1 - e^{(g-r)T}}{\sum_{i=0}^{M-1} e^{(g-r)(T-t_{i+1})}} &= N(d_1) - e^{(g-r)\Delta t} N(d_2) \\
&\quad - N(\tilde{d}_1) + e^{(u-r)\Delta t} N(\tilde{d}_2) \\
N(\tilde{d}_1) - e^{(u-r)\Delta t} N(\tilde{d}_2) &= N(d_1) - e^{(g-r)\Delta t} N(d_2) - 1 + e^{(g-r)\Delta t} \\
&= e^{(g-r)\Delta t} N(-d_2) - N(-d_1) \\
d_{1/2} &:= \frac{(r - g \pm \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}, \\
\tilde{d}_{1/2} &:= \frac{(r - u \pm \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}
\end{aligned}
$$

Interpretation

- Relation between g and u can be interpreted as follows:

> - If $e^{g\Delta t}$ is strike of a put option with a remaining contract duration Δt, so the strike price of a corresponding call option is $e^{u\Delta t}$, such that the values of these two options coincide.
>
> $\Rightarrow$ $u = g = r$ is a solution.
>
> $\Rightarrow$ For $g < r$, it holds $u > g$ due to the put-call-parity.
>
> - Generally $u(g)$ can be calculated as the solution of the above to-find-the-root with u strictly monotonically deceasing in g.

Mortality: catastrophe

- Plague in the middle ages depopulated whole countries, 40% died in certain cities
- Spanish flue costed in 1918 more lives than first world war (even though there was only a small increase in deaths at the Dutch largest insurance company in that year)
- Heat wave in Paris in August 2003 resulted in an increase of 80% of mortality for certain classes of the population
- Tsunami in Christmas 2004 whipped out whole villages in Indonesia (not insured ...)
- Some insurance companies provided insurance coverage for employees in the New York WTC

Mortality: improvements in health

- In the twentieth century health situation improved
- Most countries show decrease in mortality due to
  - ▷ Better hygiene (drinking water, maternity care)
  - ▷ Better medical care (diagnostics, therapies and medicine)
  - ▷ Less dangerous life (dull work, safe cars, less drinking/smoking)

Mortality risk: projection in the future

- For life insurance mortality projections are important
- Often insurance coverage is provided for several decades in the future
- Traditional actuarial model assumes fixed mortality rates for a certain age
- In order to include some conservatism age reductions are used in case of policies with longevity risk
- For the projection of future mortality some models have been suggested
  - ▷ With the same data you will come to different conclusions based on the different models
  - ▷ What do you model: the mortality table, a separate rate, a generation table...
  - ▷ Lee-Carter model, Milevsky and Promislov (2001)

Utility functions

- Economic agent has to make decisions under uncertainty .
- Choose "optimal" investment strategy for wealth W_0
- Decision today $\Rightarrow$ uncertain wealth W_T
- Make a trade-off between gains and losses

Maximize expected utility

- Maximize expected utility $E_P[U(X_T)]$
- By controlling optimal wealth X_T via trading strategy

$$\max_{\theta} \; E_P[U(X_T)]$$
$$\text{s.t.} \quad dX_t = (X_t r + \theta(\mu - r))dt + \theta\sigma dW_t$$

- Stochastic optimization problem
- This problem is analytically solvable for some cases
  - ▷ Merton (early 1970s)

Principle of equivalent utility

- Principle of Equivalent Utility
- Economic agent thinks about selling a (hedgeable) financial risk H_T
- Wealth at time T: $X_T - H_T$
- What price π_0 should agent ask?
- Agent will be indifferent if expected utility is unchanged:

$$\max_{X_T^*} E_P[U(X_T^*)] = \max_{X_T^{*,\pi},\pi_0} U(X_T^{*,\pi} - H_T)$$

- Principle of Equivalent Utility is consistent with arbitrage-free pricing for hedgeable claims
- Principle of Equivalent Utility can also be applied to insurance claims
  - ▷ Mixture of financial & insurance risk
  - ▷ Mixture of hedgeable & un-hedgeable risk

Pricing insurance contracts

- We have a financial (tradable) asset process S

$$dS = \mu S dt + \sigma S dW_1$$

- We have a (non-traded) insurance process y

$$dy = a dt + b(\rho dW_1 + \sqrt{1-\rho^2} dW_2)$$

  ▷ Possibly correlated with the financial process

- Assume an insurance contract with payoff at time T equal to $g(y_T)$
  - E.g. $g(y_T) = C1_{\{y<K\}}$ or more general: $g(S_T, y_T)$

Pricing insurance contract

- Solve optimal utility problem for general X_0^{π}

$$
\begin{aligned}
\max_{\theta} \quad & E[U(X_T - g(y_T))] \\
\text{s.t} \quad & dy = adt + b(\rho dW_1 + \sqrt{1-\rho^2} dW_2) \\
& dX_t = (X_t r + \theta(\mu - r))dt + \theta\sigma dW_1
\end{aligned}
$$

- Then solve for X_0^{π} such that $U_0^{*\pi}(X_0^{\pi}) = U_0^{*}(X_0)$
- Indifference-Price of contract g is then $X_0^{\pi} - X_0$
- Non-linear pde, very hard to solve in general

MuZa Solution

- Musiela and Zariphopoulou find an analytical solution for the case of exp-utility $U(x) = -1/\gamma \exp(-\gamma x)$
- For this (very special) case we have that

$$U(X - g(y)) = -1/\gamma exp(-\gamma(X - g(y))) = f(X)h(y)$$

- Exp-util has constant risk-aversion
- Risk-aversion does not depend on level of wealth X_0
- Indifference price is independent of X_0

...MuZa Solution

- MuZa find an analytical expression for the indifference price:

$$\begin{aligned}\pi(t,y) =& \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)} \ln E_t^* \left[e^{\gamma(1-\rho^2) g(y_T)} \right] \\ \approx& e^{-r(T-t)} \left(E_t^*[g(y_T)] + \frac{1}{2}\gamma(1-\rho^2) Var^*[g(y_T)] \right)\end{aligned}$$

  - ▷ Non-linear price-rule
  - ▷ Presence of non-hedgeable risks makes you risk-averse
  - ▷ This destroys additive pricing (as we know from risk-neutral pricing)
  - ▷ Also there is a difference between $+g(y)$ and $-g(y)$:
  - ▷ "seller's price" and "buyer's price"

Pricing example

- Consider specific example: life insurance contract
  - ▷ Portfolio of N policyholders
  - ▷ Survival probability p until time T
  - ▷ Pay each survivor the cash amount C
- Payoff at time T: Cn
  - ▷ n is (binomial) random variable
  - ▷ $E[n] = Np$
  - ▷ $Var[n] = Np(1-p)$

- Price of insurance contract:
  - ▷ Exponential Utility with risk aversion a
  - ▷ MuZa solution for price:

$$\pi_0 = e^{-rT}(CNp + \frac{1}{2}aC^2Np(1-p))$$

  - ▷ Variance Principle from actuarial literature.

Observations on MVM

- MVM corresponds to

$$\text{Risk-aversion}\, Var(\text{Unhedgeable Risk})$$

  - ▷ Note: MVM for "binomial risk"!

- Price for $N+1$ contracts:

$$e^{-rT}(C(N+1)p+\frac{1}{2}aC^2(N+1)p(1-p))$$

.

- Price for extra contract: . $e^{-rT}(Cp+\frac{1}{2}aC^2p(1-p))$

Observations on MVM

- Now assume that the additional $N+1$-th contract is to sell 1 contract that pays C if policyholder N dies
  - ▷ Death benefit to N's widow
  - ▷ Certain payment $C + (N-1)$ uncertain
  - ▷ Price: $e^{-rT}(C + C(N-1)p + \frac{1}{2}C^2(N-1)p(1-p))$
- Price for extra contract: $e^{-rT}(C(1-p) - \frac{1}{2}C^2 p(1-p))$
- Best estimate plus negative MVM!
- Give bonus for diversification benefit