Permanent Risk and Dynamic Incentives*

Zhiguo He† Bin Wei‡ Jianfeng Yu§

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Abstract

In many types of agency relationships the quality of the project or the agent's ability is unknown, which poses certain permanent risk to the contracting parties. We introduce such permanent risk into Holmstrom and Milgrom (1987) and derive the optimal long-term incentive contract when the agent who exerts unobservable effort faces both permanent and transitory risks. We show that the presence of permanent risk gives rise to information asymmetry because the agent who observes his own effort can distort the principal's belief by shirking. The optimal contract is designed to account for both moral hazard and asymmetric information. In Holmstrom and Milgrom (1987) with only transitory risk the optimal contract is linear. In contrast, we find that in the presence of permanent risk the optimal contract is history dependent and has an option-like feature in that incentives rise following a series of good shocks. This option-like feature curbs the agent's belief manipulation motives in earlier periods. Our model predicts that firms which are younger or have a greater degree of permanent risk tend to use more options to compensate their managers.

Keywords: Executive Compensation, Moral Hazard, Hidden Information, Continuous-Time Optimal Contracting.

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†University of Chicago, Booth School of Business, 5807 South Woodlawn Ave., Chicago 60637, Phone: (1)773-834-3769, Email: zhiguo.he@chicagobooth.edu.

‡The City University of New York, Zicklin School of Business, Baruch College, 55 Lexington Avenue, New York, NY 10010. Phone: (1)646-312-3469, Email: bin.wei@baruch.cuny.edu.

§University of Minnesota, Carlson School of Management, CSOM 3-122, 321 19th Avenue South, Minneapolis, MN 55455. Phone: (1)612-625-5498. Email: jianfeng@umn.edu.

1 Introduction

A central question in the principal-agent literature is how to provide insurance to a risk-averse agent without jeopardizing his incentives to work hard. Such trade-off between incentive provision and insurance provision has been studied by Holmstrom (1979) in a static moral hazard setting. The same idea has been extended into the classic Holmstrom and Milgrom (1987) where they consider a dynamic moral hazard setting with transitory output shocks/noises. They find that the optimal contract is linear and can be implemented by offering the agent a constant equity share. Moreover, the insurance of transitory output shocks is achieved by reducing the agent's performance share in the optimal contract. However, less attention has been paid to permanent risk, which arises, for example, if the agent has unknown ability or if the project has uncertain long-term profitability level. Little is known for the optimal working incentive and insurance provision in a dynamic moral hazard setting when both permanent and transitory risks are present.

In this paper we introduce permanent risk into the continuous-time model of Holmstrom and Milgrom (1987), and study how the permanent risk and its related endogenous dynamic learning affect optimal incentive-insurance provision in the long-term agency relations. Specifically, in our model the output process includes three components: the agent's *unobservable* effort expenditure, some transitory output shocks as in Holmstrom and Milgrom (1987), as well as a fixed parameter (e.g., the agent's ability or the project's profitability level) that is *unknown* to either the principal or the agent. Both parties learn about the unknown parameter based on the path of realized output performance under a common prior; in particular, a sequence of good (bad) realizations indicate a high (low) ability for the agent. Asymmetric information (or *hidden information*) arises endogenously because only the agent can observe his own effort and by shirking he can distort the principal's inferences. For example, if the agent chooses an effort level that is lower than is recommended, output drops accordingly, but the principal who anticipates higher effort would mistakenly attribute the drop in output to a lower value of the unknown parameter.

It is theoretically challenging to characterize the optimal incentive contract in such a dynamic agency model of both *"hidden action"* and *"hidden information"*. The main technical difficulties are due to the facts that shirking in the present setting gives rise to the aforementioned belief-manipulation effect, and that such effect is *long-lasting*. This is in sharp contrast with standard dynamic agency models where the agent's unobservable shirking has a *short-lived* effect and from each period onwards the principal knows everything (that is payoff-relevant, including his belief) regarding the agent, which allows for the simple recursive formulation of the agent's problem with the agent's continuation payoff as the only state variable.[1] This is not the case in our paper because the long-lasting belief-manipulation effect of shirking interferes with the provision of *future* incentives and the standard recursive methods are no longer applicable. Furthermore, the belief-manipulation effect also feeds back into the principal's problem since it is taken into consideration by the principal when she designs the optimal contract in the first place.

Due to these technical challenges, a dynamic agency model with persistent hidden information is in general non-tractable. In this paper, we are able to fully characterize the optimal contract in the CARA-Normal framework in Holmstrom and Milgrom (1987). We utilize the technique of calculus of variations to solve the agent's problem and then formulate the principal's problem as a free-boundary Partial Differential Equation (PDE) problem. Furthermore, we propose an innovative numerical algorithm to solve such a free boundary problem, which can be implemented by using a standard fininite-difference method. To achieve more analytical tractability, by perturbation analysis we characterize the asymptotic behavior of optimal contract when the agent's risk-aversion coefficient is sufficiently small.

We find that permanent and transitory risks have distinct impacts on the optimal trade-off between incentive provision and insurance provision. As opposed to a constant equity share as the implementation of the optimal contract in Holmstrom and Milgrom (1987) with transitory risk only, we find that in the presence of permanent risk the optimal contract has an *option*-like feature in that incentives rise following good performance (i.e., the optimal contract is convex in firm output). The intuition comes from reducing the agent's belief manipulation incentives. As explained, the agent's current shirking leads to persistent downward distortion in the principal's belief. As a result, the principal will measure the agent's performance against a lower benchmark, implying that the agent's current shirking brings a

[1] See, e.g., Spear and Srivastava (1987), and Sannikov (2008).

stream of extra compensation in the future because of the lower benchmark. These extra compensations or payoffs from belief manipulation increase with incentive slopes in the future. This is exactly how the agent extracts information rent from the principal. The amount of information rent at a given point of time depends on these future payoffs as well as his marginal utilities at these future states. Raising incentives after good performance in the optimal contract introduces a negative correlation between incentive slope and marginal utility. Therefore, under the optimal contract higher future payoffs from belief manipulation usually come when the agent's marginal utilities are lower, and hence are not as valuable for the agent. In other words, by making the optimal contract option-like, the principal can reduce information rent extracted by the agent. Ex ante, this option-like feature curbs the agent's belief manipulation incentives in earlier periods. A direct implication is that the optimal effort policy in our model is history-dependent, different from the constant effort policy in Holmstrom and Milgrom (1987).

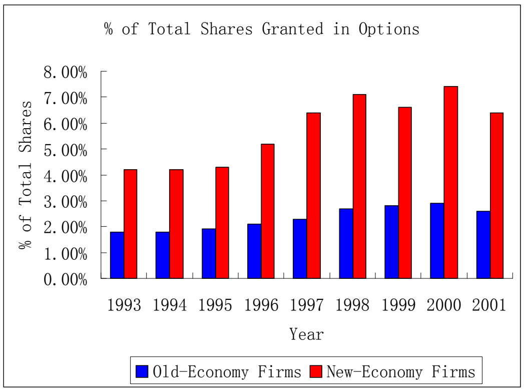

Figure 1 (A)

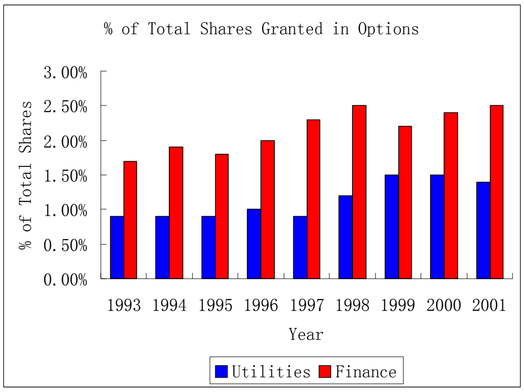

Figure 1 (B)

Furthermore, our model predicts that firms which are younger or have a greater degree of permanent risk tend to use more options to compensate their managers. Intuitively, the hidden information problem is more severe for those firms because of a larger degree of uncertainty about either the manager's ability or the firm's profitability. Therefore, the degree of convexity of the optimal contract must be higher in order to counteract with the stronger motives of the agent to shirk and manipulate the principal's belief. This prediction is consistent with the findings in Hall and Murphy (2003) that new-economy firms (such as, computer, software, the Internet, or tele-communication firms) grant more options to their CEOs than old-economy firms (Figure 1A); and financial firms that run larger human capital risk also grant more options compared to utilities firms (Figure 1B).

The main contribution of this paper is to the principal-agent literature along several important dimensions. We incorporate uncertainty and learning in the framework of Holmstrom and Milgrom (1987).[2] In a related paper, Adrian and Westerfield (2009) focus on the disagreement about the agent's ability between the principal and the agent; in that paper, the agent is dogmatic about his belief (i.e., never updates his posterior belief about profitability from past performance), which eliminates the "hidden information" problem. There, although the agent could distort the principal's belief by shirking, the dogmatic agent (who does not realize that his ability is above the one perceived by the principal) will not gain anything from this channel, and as a result there is no belief distortion effect. There are several contemporaneous papers studying similar questions. Giat and Subramanian (2010) focus on both disagreement and learning (including the agent), and derive the optimal contract under the CARA-Normal setting but in a constrained contract space with deterministic incentives. Prat and Jovanovic (2010) characterize some general properties of dynamic contracting with learning under Gaussian setting. They focus on implementing the cornered first-best constant effort, and find an increasing Pareto frontier over time in the optimal long-term contract. DeMarzo and Sannikov (2008) consider a different setting with a risk neutral agent and implementing cornered constant effort, where risk sharing is not an issue. In contrast, we allow for general interior and history-dependent effort policies, and emphasize the distinctions between permanent risk and transitory output risk in designing the optimal long-term incentive contract.

Moreover, one of the main findings in this paper that the option-like feature in the optimal contract can curb the agent's information manipulation incentives is new relative to the existing literature. Despite the fact that stock options are one of the most prominent features of CEO compensation for U.S. executives, the standard principal-agent model with constant relative risk aversion and log-normal stock prices, once calibrated to the data, seems to predict that most CEOs should not hold any stock options, as shown in Dittman and Maug (2007).[3] There are some recent theoretical developments that try to rationalize the

[2]The follow-up works on Holmstrom and Milgrom include Schattler and Sung (1993), Ou-Yang (2005), Adrian and Westerfield (2009), and He (2010a), etc.

[3]Some authors (e.g., Bebchuk and Fried (2004)) interpret the widespread use of options as rent extraction and argue that executive compensation is set by CEOs themselves rather than boards on behalf of shareholders.

use of options in CEO compensation. Based on the static framework of Holmstrom (1979) for the class of Hyperbolic Absolute Risk Aversion (HARA) utility functions, Hemmer, Kim, and Verrecchia (2000) find that for reasonable assumptions about stock price distributions and for the HARA-class of utility functions, a convex contract may be optimal only when the manager has relative risk aversion of less than one and decreasing absolute risk aversion. Their finding thus precludes the optimality of a convex contract for the CARA utility function, in direct contrast to our setting with CARA-Normal framework. Kadan and Swinkels (2008) introduce nonviability (e.g., bankruptcy) risk into Holmstrom (1979) and find that if such nonviability risk is small options dominate stock as a way to compensate the manager. Ju and Wan (2010) demonstrate in a dynamic continuous-time setting that stock options are optimal for a log-utility agent if there is a positive lower bound constraint on compensation. The main feature of this paper, permanent risk (and the associated hidden-information problem), is absent in these papers.

The rest of this paper is organized as follows. Section 2 presents the model and characterizes the optimal contract. Section 3 contains analysis of the benchmark case where pay-performance sensitivity (PPS) is constrained to be deterministic. In Section 4 we solve for the optimal contract in the general case and investigate the implications. We conclude in Section 5. All proofs are in the Appendix.

2 The Model

We formally present the model in this section. The principal hires an agent to manage a project at $t = 0$. Later we use "she" for the principal and "he" for the agent. Time is continuous with finite horizon: $[0, T]$. Both parties consume at the terminal date T, and there is no interim consumption.

The project generates a final output (cash flow) Y_T at the terminal date T. Both parties observe the evolution of output process $\{Y_t\}$, which represents the agent's contractible performance in the interval $(t, t + dt)$:

$$dY_t = (\mu_t + \theta)\, dt + \sigma dB_t.$$

Here, $\{B_t\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, μ_t is the agent's effort level, and σ is the volatility or risk in cash flows. Relative to the classic Holmstrom and Milgrom

(1987), we add a parameter θ in the expected (incremental) output to capture the agent's unknown ability or the project's unknown profitability level. For the sake of concreteness, we use the former interpretation of unknown managerial ability throughout the paper and hence refer to the permanent risk as human capital risk. Both parties cannot observe θ directly, and they will learn it along the equilibrium path under the optimal contract. At time 0, the principal and the agent share the common prior that θ is normally distributed with mean θ_0 and variance h_0:

$$\theta \sim \mathcal{N}\left(\theta_0, h_0\right).$$

Both parties can commit to the long-term relationship; we will comment on the commitment issue later in Section 3.3.3. The principal offers the agent a contract under which the agent is compensated by C_T measurable to $\mathcal{Y}_T \equiv \sigma\left\{Y_t : 0 \leq t \leq T\right\}$. We will come back to the information set shortly. For the agent, the cumulative monetary cost for his effort policy $\mu \equiv \{\mu_t\}$ is

$$G_T \equiv \int_0^T g\left(\mu_t\right) dt.$$

For simplicity, we assume $g\left(\mu_t\right) = \frac{1}{2}k\mu_t^2$, where $k > 0$ is a positive constant. The agent has a reservation utility of $\widehat{U}$ at time 0. The principal is risk neutral and the agent has a CARA preference over his terminal consumption. These utility functions for the principal and the agent are, respectively, expressed below:

$$\begin{aligned} U^p\left(Y_T - C_T\right) &= Y_T - C_T, \\ U^a\left(C_T - G_T\right) &= -\exp\left(-a\left(C_T - G_T\right)\right). \end{aligned}$$

2.1 Bayesian Learning and Effort

Recall that at time 0, the principal and the agent share the common prior: $\theta \sim \mathcal{N}\left(\theta_0, h_0\right)$. Both parties will update their beliefs based on their own information sets respectively. In this section we will analyze the learning in the dynamic relation, and its interactions with the agent's dynamic incentives to work.

Denote the principal's information set at time t by $\mathcal{Y}_t \equiv \sigma\left\{Y_s : 0 \leq s \leq t\right\}$, which is the augmented filtration generated by Y, as she can only observe the output path. Recall that the principal can only offer a compensation contract C_T that is $\mathcal{Y}_T$-measurable. The agent's information set, $\mathcal{B}_t$, is the augmented filtration generated by Y and μ, i.e., $\mathcal{B}_t = \sigma\{Y_s, \mu_s : 0 \le s \le t\}$. Compared to the principal, the agent knows more simply because he knows his actual past effort choices μ (unobservable to the principal) which might be different from the optimal effort policy $\mu^* \equiv \{\mu_t^*\}$. This distinction is important in driving our results.

2.1.1 Principal's Learning

Suppose that the optimal contract implements the effort policy μ^*, and let us consider the principal's Bayesian learning. Under the assumption that the agent indeed follows the optimal effort policy μ^*, the principal's posterior belief about θ based on her information set is fully summarized by the first two moments:

$$m_t^* \equiv \mathbb{E}\left[\theta\,|\mathcal{Y}_t, \mu^*\right] \text{ and } h_t^* \equiv \mathbb{E}\left[(\theta - m_t^*)^2\,|\mathcal{Y}_t, \mu^*\right].$$

Standard filtering argument (e.g., Theorem 12.2 in Liptser and Shiryayev (1977)) implies that

$$dm_t^* = h_t^* \frac{dY_t - (\mu_t^* + m_t^*)\,dt}{\sigma^2} \equiv \frac{h_t^*}{\sigma} dB_t^{\mu^*}, \tag{1}$$

$$h_t^* = \frac{\sigma^2 h_0}{\sigma^2 + h_0 t}, \tag{2}$$

where $B_t^{\mu^*}$ is a standard Brownian motion under the measure $\mathcal{Q}^{\mu^*}$ induced by effort μ^*:

$$dB_t^{\mu^*} = \frac{dY_t - (\mu_t^* + m_t^*)\,dt}{\sigma}. \tag{3}$$

2.1.2 Agent's Learning

If the agent follows the optimal effort policy $\{\mu_t^*\}$, then the agent's learning process is identical to the systems characterized in (1), (2), and (3). Of course, the agent may shirk from the recommended effort $\{\mu_t^*\}$ to, say, $\{\mu_t\}$. Conditional on any (possibly off-equilibrium) effort policy $\mu = \{\mu_t\}$, the agent forms his posterior belief as

$$m_t^\mu \equiv \mathbb{E}\left[\theta\,|\mathcal{Y}_t, \mu\right] \text{ and } h_t^\mu \equiv \mathbb{E}\left[(\theta - m_t^\mu)^2\,|\mathcal{Y}_t, \mu\right].$$

The superscript μ is used here to emphasize the dependence on the agent's actual effort policy μ (which the principal does not know!). Similar argument as before yields

$$dm_t^\mu = h_t^\mu \frac{dY_t - (\mu_t + m_t^\mu)\,dt}{\sigma^2} = \frac{h_t^\mu}{\sigma} dB_t^\mu, \tag{4}$$

$$h_t^\mu = \frac{\sigma^2 h_0}{\sigma^2 + h_0 t}, \tag{5}$$

where B_t^μ is a standard Brownian motion under the measure $\mathcal{Q}^\mu$ induced by effort μ:

$$dB_t^\mu \equiv \frac{1}{\sigma}\left(dY_t - (\mu_t + m_t^\mu)\,dt\right).$$

Hence, from the agent's perspective, the cash flow process has the following dynamics:

$$dY_t = (\mu_t + m_t^\mu)\,dt + \sigma dB_t^\mu. \tag{6}$$

However, from the principal's perspective, the cash flow process evolves as

$$dY_t = (\mu_t^* + m_t^*)\,dt + \sigma dB_t^{\mu^*}. \tag{7}$$

In the next section we show that the potential divergence between (6) and (7), caused by the agent's (past) effort policy, is important in understanding the agent's additional incentive to shirk.

2.1.3 Effort and Belief Distortion

The main difficulty of introducing learning into the dynamic moral hazard problem is not learning per se. Rather, the main challenge is to deal with the issue of belief manipulation: the agent, simply by shirking from the recommended effort, can distort downward the principal's belief about his ability θ. Consider the following thought experiment which is helpful in understanding later results. Suppose that at time t the agent chooses an effort level μ_t that is lower than the recommended optimal effort μ_t^*; the output thus grows lower than is expected by the principal. With learning, the principal mistakenly attributes the lower growth to a lower value of ability θ. Importantly, this stands for belief manipulation: now the principal will believe that, *in the future*, the human capital is not as high as she would believe if the agent had taken the recommended effort μ_t^*. This downward belief manipulation is beneficial to the agent, if the future optimal contract tends to provide insurance against the agent's human capital risk. In words, by shirking, the agent makes the principal believe that his human capital is low, and therefore he receives greater compensation in the future than he should.

We now formally characterize this effect. When the agent deviates from the recommended effort path μ^* by choosing effort μ, the principal's posterior mean estimate about θ is distorted, and let us denote this distortion by

$$\Delta_t \equiv m_t^\mu - m_t^*.$$

In the Gaussian framework, the variances of the posterior beliefs of both parties coincide, and for convenience we denote them by h_t (See Eqs. (2) and (5)) which is decreasing over time:

$$h_t \equiv h_t^* = h_t^\mu = \frac{\sigma^2 h_0}{\sigma^2 + h_0 t}.$$

According to (1) and (4), the increment of Δ_t is given by

$$d\Delta_t = \frac{h_t}{\sigma^2}\left(\mu_t^* - \mu_t - \Delta_t\right) dt,$$

and therefore we can solve for Δ_s to be

$$\Delta_s = \frac{h_0}{h_0 s + \sigma^2}\int_0^s \left(\mu_t^* - \mu_t\right) dt = \frac{h_s}{\sigma^2}\int_0^s \left(\mu_t^* - \mu_t\right) dt, \tag{8}$$

because initially both the principal and the agent share the same belief at the beginning, i.e., $m_0^\mu = m_0^* = m_0$ or $\Delta_0 = 0$.

As noted before, if at time t the agent chooses an effort μ_t that is lower than μ_t^*, the principal would attribute the lower output growth to a lower ability, which leads to a positive downward belief distortion Δ_t. Eq. (8) gives a clean expression for the total (potential) belief distortion up to time t: it is the cumulative past effort distortion $\mu_t^* - \mu_t$, weighted by the current uncertainty h_s. There are two noteworthy points based on Eq. (8).

First, in a standard dynamic moral hazard problem (e.g., Holmstrom and Milgrom (1987)), the agent's effort has a short-lived effect on *affecting the firm's output*. With uncertainty and learning, Eq. (8) implies that the agent's effort has a long-lasting effect on *distorting the principal's belief*. To see this, imagine that the agent shirks by ϵ only at time t (for a length of time dt), i.e.,

$$\mu_t = \mu_t^* - \epsilon, \text{ and } \mu_s = \mu_s^* \text{ for all } s \neq t.$$

Then right after time t, the agent knows that his ability is higher by $\frac{h_t}{\sigma^2}\epsilon dt$ than what the principal believes. More importantly, according to Eq. (8), this belief divergence persists at later times $s > t$, even though the agent does not shirk at time s:

$$\Delta_s = \frac{h_s}{\sigma^2}\epsilon dt \text{ for } s \geq t. \tag{9}$$

This long-lasting effect is the key to the incentive-compatibility condition identified later in Proposition 2. Furthermore, it is important to stress that this long-lasting effect makes our optimal contracting problem non-trivial.[4]

Second, Eq. (8) indicates that with greater uncertainty (i.e., larger h_0) or at an earlier time (i.e., smaller s), shirking will cause greater distortion in the principal's belief. Intuitively, when there is more uncertainty about the agent's ability θ (i.e., a larger h_s due to either a larger h_0 or a smaller s), the principal has a greater tendency to take a low realized output as evidence for low θ. Hence, if the principle indexes the agent's future compensation based on the perceived ability, then the agent will have a stronger motive to shirk when facing a greater degree of human capital uncertainty.

2.2 The Principal-Agent Problem

Suppose the principal offers the agent a contract C_T and the agent accepts it. We say that C_T implements $\mu[C]$, if $\mu[C]$ is the optimal effort policy that solves the agent's optimization problem under the contract C_T:

$$\mu[C] \in \arg\max_{\mu} \mathbb{E}\left[-\exp\left(-a\left(C_T - G_T\right)\right)\right] \tag{10}$$
$$\text{s.t. } dY_t = \left(\mu_t + m_t^{\mu}\right) dt + \sigma dB_t^{\mu}.$$

Because the contract C_T implements $\mu[C]$, in equilibrium, the principal will have the same view as the agent regarding the cash flow process.

[4] In the standard hidden-action setting, the agent's effort is short-lived in affecting the firm's output, which allows for the simple recursive formulation in which the agent's continuation payoff is the only state variable (e.g., Spear and Srivastava (1987) or Sannikov (2008)). However, in our dynamic setting, the potential belief divergence between the principal and the agent introduces the so-called "hidden information" problem where the effort has a long-lasting effect, which is in general non-tractable.

The principal's optimization problem is the following:

$$C_t^* \in \underset{C_T}{\arg\max} \mathbb{E}\left[Y_T - C_T\right] \tag{11}$$

$$\text{s.t.} \quad dY_t = \left(\mu\left[C\right] + m_t^{\mu[C]}\right) dt + \sigma dB_t^{\mu[C]}; \tag{12}$$

$$\mu\left[C\right] \text{ solves the agent's problem (10);} \tag{13}$$

$$\mathbb{E}\left[-\exp\left(-a\left(C_T - G_T\right)\right)\right] \geq \widehat{U}. \tag{14}$$

The first equation (12) describes the evolution of output performance under the principal's belief. Importantly, it embeds the principal's Bayesian updating on $m_t^{\mu[C]}$ and $dB_t^{\mu[C]}$, where the updating is based on the correct optimal effort policy taken by the agent. The next two constraints, (13) and (14), are the agent's incentive compatibility and participation constraints, respectively. The contract must attract an agent with reservation utility $\hat{U}$ at $t = 0$. As standard in the literature, this participation constraint is only enforced at time 0 when the contract is signed.

Proposition 1 below characterizes the optimal contract. It shows that solving for the optimal contract is equivalent to determining the optimal sensitivity process $\{\beta_t\}$.

Proposition 1 *A contract C_T implements effort $\{\mu_t^*\}$ and satisfies the agent's participation constraint if and only if C_T is the terminal value of a process C_t with $C_0 \geq -\frac{1}{a}\ln(-\widehat{U})$, and*

$$dC_t = g\left(\mu_t^*\right) dt + \frac{1}{2} a \beta_t^2 \sigma^2 dt + \beta_t \left(dY_t - \left(\mu_t^* + m_t^*\right) dt\right), \tag{15}$$

where β_t is a progressive-measurable process.

Remark 1 *As a standard technique, we solve for the optimal contract by writing any contract in the form of its martingale representation in (15).*[5] *Note that the martingale representation in Eq. (15), by taking out the posterior mean m_t^* of the agent's ability (according to the principal's belief) from the output dY_t, implicitly provides the agent an insurance for his human risk (similar to taking out m_1 from the output y_2 in Section 3.1.2). However, the extent of human capital insurance will depend on the time-series pattern of wealth responses $\{\beta_t^*\}$ in the optimal contract. Intuitively, a complete insurance of human capital (say*

[5] One can also write (15) in terms of integral of gross output dY_t, without taking out the posterior mean of the agent's ability m_t^*. We will come back to this alternative representation once we obtain the optimal contract in terms of $\{\beta_t^*\}$.

θ is observable) would imply that β_t^* is constant through time, as in Holmstrom and Milgrom (1987). On the other extreme, the agent who owns the entire firm—therefore without human capital insurance—has a time-t wealth of (ignoring effort problem):

$$W_t' = \mathbb{E}_t\left[\int_0^T m_s^* ds + \int_0^T \sigma dB_s^*\right] = \int_0^t m_t^* dt + m_t^* (T-t) + \int_0^t \sigma dB_s^*.$$

One can show that the wealth response to an unexpected performance shock $\sigma dB_s^* \equiv dY_t - m_t^* dt$ is h_t/h_T, which is decreasing over time.[6] Intuitively, without insurance, the agent suffers greater human capital risk in earlier periods in which most uncertainty resolves. We will come back to this point in Section 3.3.1 when discussing the implication of optimal contract.

3 The Benchmark Case: Deterministic β

In this section we constrain the working incentives β to be deterministic, which will serve as the benchmark case. We are able to solve for the optimal contract in closed form in this benchmark case (Subsection 3.2). To illustrate the main intuition in the benchmark case, we consider a two-period example in Subsection 3.1. The implications of the benchmark model are discussed in Subsection 3.3. In the next section (Section 4), we relax the constraint of deterministic β and solve for the globally optimal contract.

3.1 A Two-Period Example

This subsection aims to illustrate the main intuition of the benchmark model. The principal hires an agent to manage a project at $t=0$, and the project only lasts for two periods $t=1,2$. The output in each period is

$$y_t = \mu_t + \theta + \varepsilon_t, \text{ for } t = 1, 2.$$

Here, the first term μ_t is the agent's effort at period t. For simplicity, the agent's monetary effort cost at period t is quadratic in his effort $\mu_t^2/2$. The last term $\varepsilon_t \sim N\left(0, \sigma^2\right)$ is the output noise at period t; this risk is transitory as ε_t's are independent across periods. The new part is the uncertain parameter

[6] Note that (1) implies that $dm_t = \frac{h_t}{\sigma} dB_t^*$, therefore $\frac{dW_t'}{\sigma dB_s^*} = (T-t)\frac{h_t}{\sigma^2} + 1 = \frac{\sigma^2 + h_0 T}{\sigma^2 + h_0 t} = h_t/h_T$.

θ with a common prior of $\mathcal{N}(0, h_0)$, which captures the agent's uncertain ability. Therefore, h_0 can be interpreted the agent's human capital risk.

Note that in this setting, standing at date 0, the variance of the final output, $y_1 + y_2$, normalized by the time length 2, is (the effort μ_t's are deterministic in equilibrium)

$$\Omega \equiv \frac{Var(y_1 + y_2)}{2} = 2h_0 + \sigma^2. \tag{16}$$

Here, the human capital risk h_0 receives a weight of 2 because the agent's ability θ affects outputs in both periods.

We only consider linear contracts with deterministic coefficients on output performance; and we will verify this class of contract is indeed optimal in the continuous-time setting with a risk neutral principal. Given a compensation contract (which is paid at the end of date 2) in the form of

$$C = \alpha + b_1 y_1 + b_2 y_2, \tag{17}$$

the agent's date 0 utility is

$$\mathbb{E}_0 \left[-\exp\left(-a \left(C - \left(\mu_1^2 + \mu_2^2 \right) /2 \right) \right) \right], \tag{18}$$

while the risk neutral principal has a value of $\mathbb{E}_0 [y_1 + y_2 - C]$. Finally, the agent has a reservation utility of $\widehat{U}$ at time 0.

It is important to note that the class of linear contracts in Eq. (17) allows for the optimal insurance of the agent's human capital, thanks to Gaussian framework. As we will see shortly, the posterior of the agent's ability will be a linear combination of past output performances (here, date 1 output y_1). The optimal insurance amounts to an indexation which subtracts this linear combination from future outputs (here, date 2 output y_2), and b_1 in Eq. (17) might have already incorporated this insurance.

3.1.1 Transitory Output Risk Only

To show that the composition of risk does not matter in the optimal contract, let us first consider the case where the entire risk consists of transitory output risk. Therefore, we can imagine another setting where

$$y_t' = \mu_t + \varepsilon_t',$$

with $\varepsilon_t' \sim N(0,\Omega)$ are independent across periods, and the per period output has a variance of Ω as defined in Eq. (16). It is immediate to derive the optimal Holmstrom-Milgrom contract to be

$$Const. + \beta' (y_1 + y_2),$$

where the optimal incentive and optimal effort are given by

$$\beta' = \mu' = \frac{1}{1+a\Omega} = \frac{1}{1+a(2h_0+\sigma^2)}. \tag{19}$$

3.1.2 With Permanent Human Capital Risk

Now we study the case with permanent human capital risk. Imagine that the principal offers the contract in Eq. (19) to the agent. This contract, which ignores the date 1 information in designing date 2 contract, is inefficient standing at the beginning of $t = 2$. Given the outcome y_1, at the end of date 1 both parties know that the agent's ability is different from the date 0 prior, with an updated posterior mean of

$$m_1 = \mathbb{E}_1[\theta] = \frac{h_0}{h_0+\sigma^2}\left(y_1 - \mu'\right). \tag{20}$$

This shock stands for the agent's human capital risk from the point view of date 0, and the insurance argument would suggest to insulate the agent's compensation going forward from this risk.[7]

Suppose that we ignore the first period effort problem (i.e., assume that at date 1 the agent still takes $\mu_1 = \mu'$). Based on the above argument, the $t = 2$ contracting problem given information y_1 should be as follows. The conditional variance of the date 2 output is

$$Var(\theta + \varepsilon_2 | y_1) = \frac{h_0\sigma^2}{h_0+\sigma^2} + \sigma^2 < 2h_0 + \sigma^2 = \Omega,$$

and therefore the optimal contract should be $Const. + \beta_2 (y_2 - m_1)$ with

$$\beta_2 = \frac{1}{1+a\left(\frac{h_0\sigma^2}{h_0+\sigma^2}+\sigma^2\right)} > \beta' = \frac{1}{1+a\Omega},$$

which implies an effort of $\mu_2 = \beta_2$. Relative to the first period, learning from y_1 implies that we have a better idea of the agent's ability, therefore better information to increase the period 2 contracting efficiency. As a result, this leads to a greater incentive slope, and a higher implemented effort at $t = 2$.

[7] Another equivalent argument a la information principal (Holmstrom (1979)) would also suggest that the optimal contracting at date 2 should filter out these "lucks" due to the agent's human capital risk.

But the assumption that the agent at $t = 1$ will take the original effort $\mu_1 = \mu'$ is problematic, due to the intricate interaction between learning, insurance, and dynamic moral hazard. It is important to realize that because the output history y_1 is endogenously affected by the agent's date 1 effort choice, the agent can lower his perceived human capital m_1 by shirking at $t = 1$.[8] Knowing that the principal will provide insurance to the agent's human capital risk at $t = 2$, which amounts to taking out his perceived ability m_1 from his date 2 performance pay (recall Eq. (20)):

$$\beta_2 (y_2 - m_1) = \beta_2 y_2 - \beta_2 \frac{h_0}{h_0 + \sigma^2} y_1 + \beta_2 \frac{h_0}{h_0 + \sigma^2} \mu',$$

the agent has the motive to make $m_1 = \frac{h_0}{h_0+\sigma^2} (y_1 - \mu')$ look worse by lowering y_1. Combining with the incentive slope β_1 at $t = 1$, the effective incentive to work on date 1 output y_1 is simply

$$b_1 = \beta_1 - \beta_2 \frac{h_0}{h_0 + \sigma^2},$$

and it is easy to show that the agent will set his optimal effort at period 1 to $\mu_1 = b_1$. Therefore, given any contract

$$Const. + \beta_1 y_1 + \beta_2 (y_2 - m_1),$$

we can define the effective incentives as

$$b_1 = \beta_1 - \beta_2 \frac{h_0}{h_0 + \sigma^2}, \text{ and } b_2 = \beta_2,$$

and the contract becomes in the form of (17). The principal's problem is (note that the agent's effort $\mu_t = b_t$):

$$\max_{\alpha, b_1, b_2} \mathbb{E} \left[\sum\nolimits_{t=1}^{2} (1 - b_t)(b_t + \theta + \varepsilon_t) - \alpha \right] = \sum\nolimits_{t=1}^{2} (1 - b_t) b_t - \alpha,$$

and the constant pay α to the agent is (recall (18))[9]

$$\alpha = -\frac{b_1^2 + b_2^2}{2} + \frac{a}{2} \left[b_1^2 \left(h_0 + \sigma^2\right) + b_2^2 \left(h_0 + \sigma^2\right) + 2 b_1 b_2 h_0 \right] - \frac{\ln\left(-\widehat{U}\right)}{a}.$$

[8]From the point view of relative performance evaluation idea, in standard models the principal can obtain the agent's "luck" information from the market or other competitors. However, in our setting the measure of luck m_1 depends endogenously on the agent's date 1 effort μ_1.

[9]Given the contract $\alpha + b_1 y_1 + b_2 y_2$ and optimal effort $\mu_t = b_t$, the agent's certainty equivalent is

$$\alpha + \frac{b_1^2 + b_2^2}{2} - \frac{a}{2} Var_0 [b_1 y_1 + b_2 y_2] = \alpha + \frac{b_1^2 + b_2^2}{2} - \frac{a}{2} \left[b_1^2 \left(h_0 + \sigma^2\right) + b_2^2 \left(h_0 + \sigma^2\right) + 2 b_1 b_2 h_0 \right].$$

Then α can be derived by equating this term with the agent's outside option $-\frac{\ln\left(-\widehat{U}\right)}{a}$.

We can easily to derive the optimal solution to be

$$b_1^* = b_2^* = \frac{1}{1 + a\left(2h_0 + \sigma^2\right)} = \frac{1}{1 + a\Omega},$$

which implies a constant effort $\mu_1^* = \mu_2^* = \frac{1}{1+a\Omega}$ in the optimal contract, exactly coincides with the case of transitory output risk in (19). Here, the optimal contract can be implemented by a constant equity share, and only the total risk Ω standing at date 0—but not the risk composition of permanent/transitory components—matters!

3.2 The Continuous-Time Extension

We now solve for the optimal contract in the constrained contract space in which $\{\beta_t\}$ is deterministic. In Proposition 2 below we derive the incentive-compatibility constraint by solving the agent's problem.

Proposition 2 *For incentive compatibility, under the assumption of deterministic β_t, β_t and μ_t^* are related by*

$$\mu_t^* = \frac{1}{k}\left(\beta_t - \int_t^T \frac{h_s}{\sigma^2}\beta_s ds\right). \tag{21}$$

Finally,

$$\mathbb{E}\left[-\exp\left(-a\left(C_T - G_T\right)\right)|\mathcal{Y}_t, \mu = \mu^*\right] = -\exp\left(-a\left(C_t^* - G_t^*\right)\right). \tag{22}$$

In light of Eq. (22), the process $W_t \equiv C_t - G_t$ can be viewed as certainty-equivalent of the agent's continuation-payoff, or agent's endogenous wealth W_t that are evolving in the employment contract. From (15), the compensation C_t is composed of three components: the reimbursement for the agent's cost of effort, an insurance payment for bearing the transitory output risk, and the variable pay for incentive purposes β_t. Since β_t measures the change of the agent's wealth as a response to the unexpected performance shock, later we call β_t the agent's wealth response in the compensation contract. But keep in mind that, Eq. (21) suggests that with permanent human capital risk, the wealth response β_t no longer directly links to the agent's incentive to work.

Remark 2 *We now give the intuition for Eq. (21), which essentially gives the agent's incentive compatibility condition. Fix time t, and imagine the agent is shirking by ϵ, i.e., setting $\mu_t = \mu_t^* - \epsilon$ (and follows* *the optimal effort policy μ^* afterwards in $(t,T]$). To investigate the marginal benefit and marginal cost of shirking, we write down the instantaneous change of the agent's total wealth $W_t = C_t - G_t$, which is the change of his future compensation dC_t in (15), minus the the effort cost that he actually spent $dG_t = g(\mu_t)\,dt$:*

$$
\begin{aligned}
& dW_t = dC_t - g(\mu_t)\,dt \\
= \;& g(\mu_t^*)\,dt + \frac{1}{2}a\beta_t^2\sigma^2 dt + \beta_t\left(dY_t - (\mu_t^* + m_t^*)\,dt\right) - g(\mu_t)\,dt \\
= \;& \underbrace{[g(\mu_t^*) - g(\mu_t)]\,dt}_{\text{save effort cost}} + \underbrace{\beta_t(\mu_t - \mu_t^*)\,dt}_{\text{lower continuation-payoff}} + \underbrace{\beta_t\Delta_t dt}_{\text{distort principal's future belief}} + \frac{1}{2}a\beta_t^2\sigma^2 dt + \sigma\beta_t dB_t^\mu \qquad (23)
\end{aligned}
$$

We convert $dB_t^{\mu^*} = dY_t - (\mu_t^* + m_t^*)\,dt$ to $dB_t^\mu = dY_t - (\mu_t + m_t^\mu)\,dt$ because dB_t^μ has zero mean under the agent's information set (and recall $\Delta_t = m_t^\mu - m_t^*$). We now check the terms that are dependent on the agent's effort choice $\mu_t = \mu_t^* - \epsilon$.

The first term in Eq. (23) gives the standard effort-cost-saving benefit. The agent incurs a lower disutility cost $g(\mu_t^* - \epsilon)$ from spending a lower effort, although he is still reimbursed for the cost of optimal effort $g(\mu_t^*)$. The marginal benefit from shirking in saving his effort cost is therefore $k\mu_t^*\epsilon dt$, as $g'(\mu_t^*) = k\mu_t^*$. The second term $\beta_t(\mu_t - \mu_t^*)$ in Eq. (23) captures the fact that in dynamic contracting, shirking will lower the agent's continuation payoff. Through this effect, the agent understands that shirking will bring about a marginal cost of $\beta_t\epsilon dt$. These two effects are in the classic Holmstrom and Milgrom (1987), and combining both gives rise to the standard incentive-compatibility condition $\beta_t = k\mu_t^*$.

It is important to note that the belief distortion Δ_t in the third term (23) is endogenously related to the agent's (past) effort decisions. This belief-distortion effect, $\beta_t\Delta_t$, is new to the extant literature. Following the discussion in Section 2.1.3, the agent knows that shirking can lead to a downward distortion on the principal's (future) belief. Moreover, in some future time $s > t$, the belief divergence $\Delta_s = m_s^\mu - m_s^* > 0$ raises his future compensation by $\beta_s\Delta_s > 0$. In addition, from (9) the shirking leads to a persistent belief divergence after time t:

$$\Delta_s = \frac{h_s}{\sigma^2}\epsilon dt \text{ for } s \in [t,T]. \qquad (24)$$

As a result, the total benefit, standing at time t, is $\int_t^T \beta_t \Delta_t dt$. Plugging in (24), we find that the additional benefit of shirking at time t is (assuming $\{\beta_t\}$ is a deterministic policy that in fact holds in the optimal contract derived later):

$$\left(\int_t^T \frac{h_s}{\sigma^2}\beta_s ds\right)\epsilon dt.$$

Combining these three effects together, the optimality of equilibrium effort implies that the total marginal benefit equals the marginal cost, i.e.,

$$k\mu_t^* = \beta_t - \int_t^T \frac{h_s}{\sigma^2}\beta_s ds. \tag{25}$$

This is exactly the incentive-compatibility condition identified in (21). Therefore, the effective incentive provision is the wealth response β_t, minus the weighted average of future wealth responses $\{\beta_s\}_{s\in[t,T]}$. We will come back to this point when discussing the implementation of optimal contract in Section 3.3.1.

Note that Proposition 2 allows us to replace the incentive compatibility and participation constraints in Eq. (13) and Eq. (14) in the principal's problem by Eq. (15) and Eq. (21). Essentially, Proposition 2 establishes an important link between the implemented effort $\{\mu_t^*\}$ and the wealth response $\{\beta_t\}$ in any incentive-compatible contracts, which allows the principal to choose $\{\beta_t\}$ (in turn choose $\{\mu_t^*\}$) to maximize her value. Finally, once we find the optimal wealth response $\{\beta_t^*\}$, the optimal contract $\{C_t^*\}$ and optimal effort $\{\mu_t^*\}$ are determined by (15) and (21), respectively. Therefore we can write the principal's problem as

$$\beta^* \in \arg\max_{\beta} \mathbb{E}\left[Y_T - C_T\right] \tag{26}$$

$$s.t. \quad dY_t = (\mu_t + m_t^\mu)dt + \sigma dB_t^\mu, \tag{27}$$

$$dm_t^\mu = \frac{h_t}{\sigma} dB_t^\mu, \tag{28}$$

$$dC_t = g\left(\mu_t\right)dt + \frac{1}{2}a\beta_t^2\sigma^2 dt + \beta_t \sigma dB_t^\mu, \text{ with } C_0 = -\frac{1}{a}\ln(-\widehat{U}), \tag{29}$$

$$\mu_t = \frac{1}{k}\left(\beta_t - \int_t^T \frac{h_s}{\sigma^2}\beta_s ds\right). \tag{30}$$

Here, (27) and (28) are from Bayesian updating; (29) and (30) are from Proposition 2. We have invoked the binding participation constraint in (29) in the optimal contract by setting $C_0 = -\frac{1}{a}\ln(-\widehat{U})$. Substituting the constraints into the principal's objective function, we can simplify the principal's maximization problem into the following problem:

$$\max \int_0^T \left(\mu_t - \frac{k}{2}\mu_t^2 - \frac{a\sigma^2}{2}\beta_t^2 \right) dt$$
$$s.t. \quad \mu_t = \frac{1}{k}\left(\beta_t - \int_t^T \frac{h_s}{\sigma^2}\beta_s ds \right). \tag{31}$$

Notice that (31) introduces a forward-looking component in the constraint, which in general makes the problem (26) intractable. More importantly, we cannot use the backward induction argument to solve for

$$\left\{ \beta_T^*, \beta_{T-dt}^*, \beta_{T-2dt}^*, \cdots \right\}$$

successively step by step. It is because in choosing the last wealth response β_T^* at time T, the principal should take into account its impact on the choice of earlier wealth response β_t^*'s. Backward induction procedure will miss this effect and lead to a higher final wealth response β_T^* than it should be. We use variational method to derive the solution in closed form. The following theorem gives one of the main results of this paper.

Theorem 1 *(i) Define the total risk per unit of time as*

$$\Omega \equiv \frac{Var_0\left[Y_T\right]}{T} = Th_0 + \sigma^2.$$

In the optimal contract, the implemented optimal effort only depends Ω and is given by

$$\mu_t^* = \frac{1/k}{1 + ak\Omega} \equiv \mu^*. \tag{32}$$

The optimal wealth response $\{\beta_t^*\}$ is given by:

$$\beta_t^* = \frac{h_t}{h_T}\frac{1}{1 + ak\Omega} = \frac{h_t}{h_T}k\mu^*, \tag{33}$$

(ii) Denote $p_t \equiv \int_t^T \frac{h_s}{\sigma^2}\beta_s ds$. Then along the equilibrium path,

$$p_t^* = \frac{h_t/h_T - 1}{1 + ak\Omega} = \frac{h_t\left(T - t\right)}{h_0 T}p_0^*, \tag{34}$$
$$\beta_t^* = \frac{h_t}{h_T}\frac{1}{1 + ak\Omega} = \frac{\sigma^2}{\left(T - t\right)h_T}p_t^*. \tag{35}$$

(iii) At time t and given $p_t = p$, along a possibly off-equilibrium path (i.e., p is arbitrary and may not equal p_t^*), the principal's value function is given by

$$V^*(t,p) = -\frac{1}{2}A_t p^2 + B_t p, \tag{36}$$

where

$$A_t = \frac{\sigma^4}{(T-t)\,h_t}\left(\frac{1}{kh_t} + \frac{a\sigma^2}{h_T}\right) \text{ and } B_t = \frac{\sigma^2}{kh_t}. \tag{37}$$

Moreover, starting from (t,p), for $s \in [t,T]$, it must be true that $\beta_s = -\frac{\sigma^2}{h_t}p_s'$ and

$$p_s = \frac{h_s/h_T - 1}{h_t/h_T - 1}p. \tag{38}$$

It is worthwhile to note that when h_0 goes to 0, it reduces to the original Holmstrom-Milgrom problem.[10] As in the two-period example, the optimal effort policy turns out to be constant in the constrained contract space with deterministic incentives. This surprising result appears to be identical to that in Holmstrom and Milgrom (1987), however, the underlying mechanisms are totally different. In Holmstrom and Milgrom (1987), the optimal effort policy being constant comes from the fact that model is set in a *stationary* environment without permanent human capital risk. By contrast, given that the unknown parameter is fixed, learning improves over time and thus the insurance for permanent risk will be time-dependent in our paper. Specifically, the insurance against the permanent risk is provided by indexation of future wages, that is, the firm may provide *future* insurance for *earlier periods*' low output performance, as this represents bad realization of the permanent risk. In our paper the improved information or contracting efficiency seems to induce increasing effort levels over time.

However, there is another counteracting force at play, which is rooted in the potential belief manipulation incentives. In our model, the agent will have additional shirking incentives in earlier periods to

[10] In fact,

$$\begin{aligned}
\lim_{h_0\to 0} B_t p_t^* &= \lim_{h_0\to 0}\frac{\sigma^2}{k(1+ak\Omega)}\frac{h_t/h_T - 1}{h_t} = \frac{\sigma^2}{k(1+ak\Omega)}\frac{\lim_{h_0\to 0} d(h_t/h_T)/dh_0}{\lim_{h_0\to 0}(dh_t/dh_0)} = \frac{T-t}{k(1+ak\Omega)} \\
\lim_{h_0\to 0} A_t (p_t^*)^2 &= \frac{\sigma^4}{(T-t)}\left(\frac{1}{k}\lim_{h_0\to 0}\frac{(p_t^*)^2}{h_t^2} + a\sigma^2 \lim_{h_0\to 0}\frac{p_t^*}{h_T}\lim_{h_0\to 0}\frac{p_t^*}{h_t}\right) = \frac{T-t}{k(1+ak\Omega)}
\end{aligned}$$

Therefore $\lim_{h_0\to 0} V^*(t,p_t^*) = \frac{T-t}{2k(1+ak\Omega)}$, which is the maximum of the principal's expected utility in Holmstrom and Milgrom (1987).

manipulate downward his ability perceived by the principal. Effectively, offering insurance to the permanent risk works in the manner that the principal indexes the agent's future performance against the perceived ability level, i.e., only compensates for the agent's unexpected performance conditional on the date t information. However, the date t information, which consists of the output performance history, is endogenously controlled by the agent. The better the output history, the higher the ability perceived by the principal. As a result, the agent, knowing that in the future the principal tends to take out the perceived ability level from the output, will want to shirk in earlier periods in order to downward distort the principal's belief. In other words, providing insurance in the future makes the agent lazy in early periods. The higher the future pay-performance incentives, the greater the information manipulation incentives. Therefore, as a response, the principal lowers the future incentives to reduce the agent's information manipulation incentives in earlier periods. Therefore, this mechanism predicts that the implemented effort in the optimal contract should be decreasing over time.

We show that these two opposite forces ("contracting efficiency" and "belief manipulation") exactly cancel out with each other in the benchmark case when the pay-performance incentives are constrained to be deterministic. This makes the permanent risk *almost* identical to the transitory output risk in the benchmark case. As a result, in the (locally) optimal contract within the constrained contract space, all payoff-relevant predictions, including the implemented effort level, risk sharing profile, and both parties' time-zero values, depend only on the total variance as a sum of (date 0) permanent human capital risk and transitory output risk, not the specific composition of these two risk categories. This implies that given the consideration of dynamic incentive provision, the optimal contract offers very limited insurance for permanent human capital risk, and effectively the principal refuses to learn/update the agent's unknown ability.

3.3 Discussions

3.3.1 Implementation of optimal contract: constant equity share

As shown in the following corollary of Theorem 1, our optimal contract can be implemented by a constant equity share b^* which dictates the sharing rule between the principal and the agent for the output dY_t

at period t. This result, which coincides with Holmstrom and Milgrom (1987), immediately implies a constant pay-performance sensitivity over time in the optimal contract. The corollary is essentially an alternative representation of the optimal contract in terms of the output history $\{dY_t : 0 < t < T\}$. Note that the incentive contract in (15) takes out the posterior mean m_t^*, which is effectively a linear combination of past output history $\{dY_s : 0 < s < t\}$, from current output dY_t. Then the result follows by simply expanding m_t^* in terms of $\{dY_s : 0 < s < t\}$, and regrouping these outputs.

Corollary 1 *The optimal contract can be implemented by a constant equity share b^* (which is also pay-performance sensitivity):*

$$b^* = k\mu^* = \frac{1}{1 + ak\Omega},$$

and a constant wage:

$$\alpha^* = -\frac{1}{a}\ln(-\widehat{U}) - T\left[\theta_0 b^* + \frac{(b^*)^2}{2k} - \frac{a}{2}(b^*)^2\,\Omega\right].$$

In other words, the optimal contract can be written as

$$C^* = \alpha^* + b^* Y_T.$$

Proof. *We need to rewrite the contract in Eq. (15) in terms of output dY_t directly. To this end, we can work out posterior m_t^* as a linear combination of output history:*

$$dm_t^* = \frac{h_t}{\sigma^2}\left(dY_t - \mu^* dt - m_t^* dt\right) \Rightarrow m_t^* = \frac{h_t}{\sigma^2}\int_0^t \left(dY_s - \mu^* ds\right).$$

Therefore,as an alternative representation of (15), the agent's terminal wealth can be rewritten as

$$\begin{aligned} W_T &= C_T - G_T = C_0 + \int_0^T \frac{1}{2}a\beta_t^{*2}\sigma^2 dt + \int_0^T \beta_t^*\left(dY_t - \mu^* dt - m_t^* dt\right) \\ &= C_0 + \int_0^T \frac{1}{2}a\beta_t^{*2}\sigma^2 dt + \int_0^T \left[\beta_t^* - \int_t^T \beta_s^*\frac{h_s}{\sigma^2}ds\right]\left(dY_t - \mu^* dt\right), \end{aligned}$$

where we use integration-by-parts to express $\int_0^T \beta_t^* m_t^* dt$ in terms of $\{dY_s\}$. Under this (equivalent!) representation without indexation of posterior mean m_t^*, the loading

$$\beta_t^* - \int_t^T \beta_s^*\frac{h_s}{\sigma^2}ds$$

is just the agent's effective incentive to spend effort, as suggested by (21). As a result, if we define the performance share as[11],

$$b^* \equiv \beta_t^* - \int_t^T \beta_s^* \frac{h_s}{\sigma^2} ds = k\mu^* = \frac{1}{1 + ak\Omega},$$

then the optimal contract can be implemented by giving the agent an inside share of $b^* = k\mu^*$. *The calculation of constant wage* α^* *is obvious.* ■

This corollary has an important implication. The diminishing wealth response β_t^* over time in Theorem 1 seems to suggest that when time goes by the agent's pay-performance sensitivity (so the risk borne by the agent) goes down, which in turn implies that the risk sharing profile improves over time. This reasoning is flawed. In a learning environment with constant unknown ability θ, the volatility tends to be greater in earlier periods as there are more uncertainty resolved in the beginning. The declining of β_t^* itself is a symptom of the lack of insurance of human capital risk, because a full insurance implies that β_t^* to be constant through time as in Holmstrom and Milgrom (1987) (see related discussion after Proposition 2.) In fact, the volatility of firm value is also declining over time. Then the empirical pay-performance sensitivity (e.g., Jensen and Murphy (1992)) should be measured as a ratio between the responses of the agent's wealth and of the firm's value. Once we use this definition, the agent's pay-performance sensitivity is just the constant equity share b^* which is no longer time varying.

3.3.2 Irrelevance of risk composition

More interestingly, we find that the optimal constant equity share b^* and the optimal constant effort μ^* only depend on the total risk per unit of time standing at initial date $t = 0$:

$$\Omega = Th_0 + \sigma^2,$$

which is independent of the risk composition of permanent human capital risk h_0 or transitory output risk σ^2. Furthermore, the risk composition is also irrelevant in determining both parties' payoffs under the optimal contract, because the principal and the agent are holding some constant shares of the project whose final output Y_T has a total variance of $T\Omega$.

[11]Note that $dh_s = -\frac{1}{\sigma^2} h_s ds$, therefore $\beta_t^* - \int_t^T \beta_s^* \frac{h_s}{\sigma^2} ds = \frac{h_t}{h_T} k\mu^* - \frac{k\mu^*}{h_T} \int_t^T \frac{h_s^2}{\sigma^2} ds = \frac{h_t}{h_T} k\mu^* + \frac{k\mu^*}{h_T} (h_T - h_t) = k\mu^*$.

One particular interesting (and perhaps also surprising) limiting case would be letting the transitory output risk σ^2 go to zero (but not exactly zero). There, the agent's ability becomes known right after the employment. As a result, looking forward, *there are no future noises and therefore no contracting frictions in the long-term employment relation*. However, surprisingly, in the optimal contract the agent still holds a constant share of the firm and exerts a constant (relatively low) effort throughout the employment history. This case vividly illustrates two important points. First, we can have the lack of insurance for human capital risk, even though the agent's ability can be almost known for sure later on. Second, the continuation contract, although is optimal standing at $t = 0$, can be seemingly (extremely) inefficient.[12]

3.3.3 Agent's human capital v.s. firm's profitability uncertainty: Commitment issue

Throughout the paper we have interpreted the uncertain parameter θ as the agent's ability, or his human capital. The whole analysis still goes through if we equivalently interpret θ as the uncertain profitability of the firm/project, and the same irrelevance result characterized in Theorem 1.

However, economically, there is one salient distinction between human capital and firm's profitability. In the influential paper by Harris and Holmstrom (1982), the agent cannot commit to stay within the long employment relationship (as he can work at another identical firm). As a result, the full (first-best) insurance of human capital is infeasible, and when the agent's human capital turns out to be sufficiently high ex post, the firm adjusts the agent's wage upward to prevent him from leaving. This one-sided commitment issue is much less of a concern for the project's uncertain quality.

It is clear in our model the commitment is crucial. Indeed, in our model the potential ex post contracting efficiency (or Pareto efficiency) improves over time when the agent's ability becomes more and more certain, but the optimal contract binds both parties to move outward. However, this commitment, which requires both parties not to renegotiate the continuation contract, is different from the agent's one-sided commitment in Harris and Holmstrom (1982).[13] In our model, because of the dynamic agency issue, the

[12] The same idea can also be seen in He (2010b) where the agent can privately save in a long-term dynamic agency relation. In the optimal contract derived in that paper, the agent who is fired for his past poor performance will be granted a generous severance package.

[13] For instance, in contrast of competitive firms as in Harris and Holmstrom (1982), suppose that we (counterfactually, of course) assume agents (independent of the level of ability) are competitive, so that firms enjoy all the rent. Then the one-sided commitment has no bite on our model. However, the optimal long-term contract in our model still suffers the commitment issue

insurance of human capital risk is very limited, and looking forward the agent receives a constant share of his human capital always. For future work, it will be interesting to combine our model with Harris and Holmstrom (1982) to see how the market force in the labor market affects the long-term incentive contracts.

4 The General Case

In this section, we relax the assumption of deterministic β_t and solve for the optimal contract with possibly stochastic β_t. In the rest of the paper, without loss of generality, we assume $k = 1$ to save notation.

4.1 The Optimal Contract

We first relax the assumption of β_t being deterministic, and then use a variational argument following Cvitanic, Wan, and Zhang (2009) to solve the agent's problem as the incentive-compatibility constraint. Similar results are obtained in Prat and Jovanovic (2010).

Proposition 3 *The incentive compatibility constraint in the agent's problem is given by*

$$\mu_t = \beta_t - \mathbb{E}_t^{\mu}\left[\int_t^T \frac{h_s\beta_s}{\sigma^2} e^{\left(-\int_t^s a\beta_v \sigma dB_v^{\mu} - \frac{1}{2}\int_t^s a^2\beta_v^2\sigma^2 dv\right)} ds\right]. \tag{39}$$

Comparing the above expression with Eq. (31) when $\{\beta_t\}$ is assumed to be deterministic, there exists an additional term $e^{\left(-\int_t^s a\beta_v \sigma dB_v^{\mu} - \frac{1}{2}\int_t^s a^2\beta_v^2\sigma^2 dv\right)}$ in the incentive-compatibility constraint. If we define the agent's continuation utility at time t by $\mathcal{V}_t$ from following the recommended effort $\{\mu_s\}_{s\geq t}$, i.e.,

$$\mathcal{V}_t \equiv \mathbb{E}_t\left[-\exp\left[-a\left(C_T - G_T\right)\right]|\mathcal{Y}_t, \mu\right],$$

then this additional term can be shown to equal $\mathcal{V}_s/\mathcal{V}_t$.[14] As a well-known property of exponential utility, i.e., the utility level is proportional to the marginal utility, this term is equal to the ratio of marginal

which basically says that both parties would like to rewrite the contract when time passes by.

[14]In fact, given effort $\{\mu_t\}$ to implement, the agent's continuation utility is given by

$$\mathcal{V}_t \equiv \mathbb{E}_t\left[-\exp\left[-a\left(C_T - G_T\right)\right]|\mathcal{Y}_t, \mu\right]$$

By definition of the $\{C_t\}$ process, $\mathcal{V}_t = -\exp\left[-a\left(C_t - G_t\right)\right]$ which evolves as

$$d\mathcal{V}_t = a\exp\left[-a\left(C_t - G_t\right)\right]\left[dC_t - g\left(\mu_t\right)dt - \frac{1}{2}a\sigma^2\beta_t^2 dt\right] = -a\sigma\beta_t\mathcal{V}_t dB_t^{\mu}$$

utilities. This additional term indicates that when $\{\beta_t\}$ is a stochastic process, the belief-manipulation effect on the agent's incentive to shirk varies with his future marginal utilities. With risk-averse agent, the current belief-manipulation gain goes down if the future gains are greater exactly in the lower marginal utility states. When $\{\beta_t\}$ is constrained to a deterministic process in the benchmark case, the future gains are independent of the agent's marginal utilities. However, by making β_t stochastic, the principal can control the extent to which the agent's future gains from shirking today to manipulate her belief co-vary with his marginal utilities. In this way, the principal can mitigate the belief-manipulation effect.

We now solve the optimal contract given the incentive-compatibility constraint in (39). We first write the problem in a recursive form which allows for standard dynamic programming technique. The Exponential-Normal framework allows us to write the problem with two state variables, with one being time index t. The other state variable captures the future information rent, is at the core of the problem:

$$p_t \equiv \mathbb{E}_t \left[\int_t^T \frac{h_s}{\sigma^2} \beta_s \exp\left(-\int_t^s a\beta_u \sigma dB_u - \int_t^s \frac{1}{2} \left(a\beta_u \sigma\right)^2 du \right) ds \right].$$

Note that the dynamics of the state variable p_t can be characterized as follows

$$dp_t = \beta_t \left(a\sigma\sigma_t^p - \frac{h_t}{\sigma^2} \right) dt + \sigma_t^p dB_t \tag{40}$$

With this definition, the principal's problem is

$$\begin{aligned} &\max_{\beta,\sigma^p} \mathbb{E} \left[\int_0^T \left(\mu_t - \frac{\mu_t^2}{2} - \frac{a\sigma^2}{2}\beta_t^2 \right) dt \right] \\ &s.t. \quad \mu_t = \beta_t - p_t \end{aligned}$$

Define the principal's value function at time t, as a function of time t and future information rent p, as

$$\begin{aligned} V(t,p) \quad &= \quad \max_{\beta,\sigma^p} \mathbb{E}_t \left[\int_t^T \left(\mu_s - \frac{\mu_s^2}{2} - \frac{a\sigma^2}{2}\beta_s^2 \right) ds \right] \\ &s.t. \;\; \mu_s = \beta_s - p_s, \\ &p_t = p, p_T = 0. \end{aligned} \tag{41}$$

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or

$$d\log(-\mathcal{V}_t) = -\frac{1}{2}a^2\sigma^2\beta_t^2 dt - a\sigma\beta_t dB_t^\mu \Leftrightarrow \frac{\mathcal{V}_s}{\mathcal{V}_t} = \exp\left[-\int_t^s a\beta_v \sigma dB_v^\mu - \frac{1}{2}\int_t^s a^2\beta_v^2\sigma^2 dv \right]$$

Then the principal's problem can be written in the following recursive form

$$\begin{aligned} V(0,p) &= \max_{\beta,\sigma^p} \mathbb{E}\left[\int_0^t \left(\mu_t - \frac{\mu_t^2}{2} - \frac{a\sigma^2}{2}\beta_t^2\right)dt + V(t,p_t)\right] \\ & s.t.\ \ \mu_s = \beta_s - p_s, p_0 = p. \end{aligned}$$

The underlying reason for this recursive formulation is that the optimal continuation contract after t derived under (41) can be used as a part of optimal contract standing at time $s < t$, as long as the continuation contract fulfills the forward looking constraint in $\mu_s = \beta_s - p_s$.[15]

Using the standard dynamic programming technique, we have the principal's HJB equation as

$$0 = \max_{\beta,\sigma^p} (\beta - p) - \frac{1}{2}(\beta - p)^2 - \frac{a\sigma^2}{2}\beta^2 + V_p\beta\left(\sigma^p a\sigma - \frac{h_t}{\sigma^2}\right) + \frac{1}{2}V_{pp}(\sigma^p)^2 + V_t. \tag{42}$$

with boundary conditions

$$V(T,0) = 0, \text{ and } V(T,p) = -\infty \text{ for } p \neq 0.$$

The associated first-order conditions are

$$\beta(t,p) = \frac{1 + p - \frac{h_t}{\sigma^2}V_p}{1 + a\sigma^2 + a^2\sigma^2\frac{V_p^2}{V_{pp}}}, \tag{43}$$

$$\sigma^p(t,p) = -a\sigma\beta\frac{V_p}{V_{pp}}. \tag{44}$$

In the general case with possibly stochastic β_t, we prove in the following proposition that there exist a lower boundary $p \equiv 0$ and an upper boundary $\overline{p}_t \geq 0$ such that without loss of generality we can focus

[15]The major concern is that the choice of $\{\beta_u\}_{u\geq t}$ as a solution to the optimal continuation contract following time t affects the incentive constraint $\mu_s = \beta_s - p_s$ before t, as p_s is forward looking. However,

$$\begin{aligned} p_s &= \mathbb{E}_s\left[\int_s^T \frac{h_v}{\sigma^2}\beta_v \exp\left(-\int_s^v a\beta_u\sigma dB_u - \int_s^v \frac{1}{2}(a\beta_u\sigma)^2 du\right)dv\right] \\ &= \mathbb{E}_s\left[\int_s^t \frac{h_v}{\sigma^2}\beta_v \exp\left(-\int_s^v a\beta_u\sigma dB_u - \int_s^v \frac{1}{2}(a\beta_u\sigma)^2 du\right)dv\right] \\ &\quad + \mathbb{E}_s\left[\begin{array}{c} \exp\left(-\int_s^t a\beta_u\sigma dB_u - \int_s^t \frac{1}{2}(a\beta_u\sigma)^2 du\right)\cdot \\ \int_t^T \frac{h_v}{\sigma^2}\beta_v \exp\left(-\int_t^v a\beta_u\sigma dB_u - \int_t^v \frac{1}{2}(a\beta_u\sigma)^2 du\right)dv \end{array}\right] \\ &= \mathbb{E}_s\left[\int_s^t \frac{h_v}{\sigma^2}\beta_v \exp\left(-\int_s^v a\beta_u\sigma dB_u - \int_s^v \frac{1}{2}(a\beta_u\sigma)^2 du\right)dv\right] \\ &\quad + \mathbb{E}_s\left[\exp\left(-\int_s^t a\beta_u\sigma dB_u - \int_s^t \frac{1}{2}(a\beta_u\sigma)^2 du\right)\cdot p_t\right] \end{aligned}$$

where the last equality uses the law of iterated expectations. Therefore, $\{\beta_u\}_{u\geq t}$ enters the incentive constraint at s only through p_t, and the optimal continuation contract which repect p_t will not affect the agent's incentive before t.

on the meaningful region $p \in [0, \overline{p}_t]$ at date t. Furthermore, Proposition 4 below shows that the lower boundary $p \equiv 0$ is an absorbing one in the sense that once it is hit the state variable $\{p_t\}$ will remain to be zero from hitting time on and the agent is asked to spend zero effort. That is, the lower boundary is the firing boundary whenever it is hit the agent will be fired. However, we can show that in equilibrium, the agent will never be fired.[16] On the other hand, the upper boundary $p = \overline{p}_t$ along which $V_p(t, \overline{p}_t) = 0$ is satisfied is a reflecting upper boundary. That is, along the equilibrium path, at any point of time t, p_t always stays below $\overline{p}_t$.

Proposition 4 *If we denote by $p = \overline{p}_t$, $t \in [0, T]$ the boundary such that $V_p(t, \overline{p}_t) = 0$ for any t. Then once this boundary is hit, say at time t, then $(s, p(s))$ for $s \geq t$ is always below this boundary. That is, this boundary is a reflecting one. In addition, $p \equiv 0$ is an absorbing boundary.*

Together with the concavity of $V(t, \cdot)$ (i.e., $V_{pp}(t, p) < 0$ for a given t), it follows from Proposition 4 that in the optimal contract $\{\beta_t\}$ initially at time 0 the principal sets $p_0 = \overline{p}_0$, and afterwards the state variable p_t evolves according to Eq. (40).

4.2 Asymptotic Analysis For a Risk-Tolerant Agent

To achieve more analytical tractability, we now utilize an asymptotic analysis to explore the implications of the optimal contract when the agent is sufficiently risk tolerant, i.e., a is small enough.

When the agent is risk neutral (i.e. $a = 0$), it is straightforward to check that the deterministic policy is indeed optimal. The value function in this case is denoted by $V^o(t, p)$

$$V^o(t, p) = -\frac{1}{2} A_t^o p^2 + B_t^o p, \tag{45}$$

where $A_t^o = \frac{\sigma^4}{(T-t)h_t^2}$ and $B_t^o = \frac{\sigma^2}{h_t}$. The optimal upper boundary is given by

$$\overline{p}_t^o \equiv \frac{B_t^o}{A_t^o} = \frac{h_t(T-t)}{\sigma^2} = \frac{T-t}{t + \sigma^2/h_0}.$$

The upper boundary in the case of risk-neutral agent is an absorbing one.

[16]We can prove that when p gets close to zero, the dynamics of p becomes asymptotically similar to that of a geometric Brownian motion. The proof, although not reported in the text, is available upon request.

In Proposition 5 we report the results of the asymptotic analysis which expands the value and policy functions around those in the case of risk-neutral agent.

Proposition 5 *When a is small enough, we have the following second-order approximations:*

$$\overline{p}_t = \overline{p}_t^o - aq_1(t) - a^2 q_2(t) + O\left(a^3\right), \tag{46}$$

$$V(t,p) = V^o(t,p) + aH_1(t)p^2 + a^2 H_2(t)\left(p - \overline{p}_t^o\right)^2 p^2 + O\left(a^3\right). \tag{47}$$

Furthermore, we show that

$$\beta(t,p) = p\left(\frac{T+\sigma^2/h_0}{T-t} + a^2 N_1(t,p)\right) + O\left(a^3\right), \tag{48}$$

$$\sigma^p(t,p) = -a\sigma\frac{T+\sigma^2/h_0}{T-t}p\left(p-\overline{p}_t\right) + O\left(a^3\right), \tag{49}$$

where the functional forms of $q_1(t) > 0 > q_2(t)$, $H_1(t) < 0 < H_2(t)$ and $N_1(t,p) > 0$ are given in the proof in the Appendix.

For notational simplicity, we will omit the higher-order terms $O\left(a^3\right)$ from now on. One immediate implication from Proposition 5 is that both $\beta(t,p)$ and $\sigma^p(t,p)$ are non-negative within the domain $p \in [0, \overline{p}_t]$ for any given t. That is, at time t, for $p \in [0, \overline{p}_t]$,

$$\beta(t,p) \geq 0 \text{ and } \sigma^p(t,p) \geq 0. \tag{50}$$

From the expression of $H_1(t,p)$ in Eq. (75) in the Appendix and the observation that $A_t = A_t^o + aH_1(t)$, it follows $V^o(t,p) + aH_1(t)p^2 = V^*(t,p)$. Therefore, Eq. (47) shows that under the optimal contract the value function improves by the magnitude of order 2. The similar result also holds for incentives from Eq. (48). Specifically,

$$V(t,p) - V^*(t,p) = a^2 H_2(t)\left(p-\overline{p}_t^o\right)^2 p^2 \geq 0,$$

$$\beta(t,p) - \beta^*(t,p) = a^2 p N_1(t,p) \geq 0,$$

where $\beta^*(t,p) \equiv \frac{T+\sigma^2/h_0}{T-t}p$ is derived from Theorem 1(iii). We can also show that the second-order approximation of $\overline{p}_t$, i.e., $\overline{p}_t^o - aq_1(t) - a^2 q_2(t)$, is strictly lower then B_t/A_t but only by the magnitude of order 3. Figures 2-5 plot the upper boundary $\overline{p}_t$, the value function, and optimal policies $\beta(t,p)$ and $\sigma^p(t,p)$, respectively.

Furthermore, from Eqs. (48) and (49), the dynamics of $\{p_t\}$ is given by

$$\begin{aligned} dp_t &= \beta(t,p_t)\left(a\sigma\sigma^p(t,p_t)-\frac{h_t}{\sigma^2}\right)dt+\sigma^p(t,p_t)\,dB_t \\ &= p\left(\frac{T+\sigma^2/h_0}{T-t}+a^2N_1(t,p)\right)\left(-a^2\sigma^2\frac{T+\sigma^2/h_0}{T-t}p\,(p-\overline{p}_t)-\frac{h_t}{\sigma^2}\right)dt \\ &\quad -a\sigma\frac{T+\sigma^2/h_0}{T-t}p\,(p-\overline{p}_t)\,dB_t \\ &= p\left[-\frac{T+\sigma^2/h_0}{(T-t)(t+\sigma^2/h_0)}+a^2\underbrace{\left(-\sigma^2\left(\frac{T+\sigma^2/h_0}{T-t}\right)^2p\,(p-\overline{p}_t)-\frac{h_tN_1(t,p)}{\sigma^2}\right)}_{\equiv N_2(t,p)\leq 0}\right]dt \\ &\quad -a\sigma\frac{T+\sigma^2/h_0}{T-t}p\,(p-\overline{p}_t)\,dB_t \\ &\equiv p\left[-\frac{T+\sigma^2/h_0}{(T-t)(t+\sigma^2/h_0)}+a^2N_2(t,p)\right]dt-a\sigma\frac{T+\sigma^2/h_0}{T-t}p\,(p-\overline{p}_t)\,dB_t. \end{aligned} \tag{51}$$

It follows from the above equation that under the optimal contract, p_t has a more negative drift, but also has a non-negative diffusion term.

Remark 3 *We now illustrate the option-like feature of the optimal contract. By Ito's lemma*

$$d\beta=\beta_t dt+\beta_p dp_t+\frac{1}{2}\beta_{pp}(dp_t)^2.$$

From Eqs. (48) and (51), we have

$$d\beta=\left(\beta_t+\beta_p p\left[-\frac{T+\sigma^2/h_0}{(T-t)(t+\sigma^2/h_0)}+a^2N_2(t,p)\right]\right)dt-\underbrace{\frac{a\sigma\left(T+\sigma^2/h_0\right)^2}{(T-t)^2}p\,(p-\overline{p}_t)}_{\geq 0}dB_t.$$

The diffusion term of $d\beta(t,p)$ is equal to $\frac{T+\sigma^2/h_0}{T-t}\sigma^p(t,p)\geq 0$. Therefore, after realizations of good shocks, the incentives $\beta(t,p)$ increase. This effect is very similar to that of options: Good shocks increase stock prices, which in turn increases the delta (or the hedge ratio) of an European call option.

Furthermore, the degree of the option-like feature of the optimal contract is tied to the diffusion term of $d\beta(t,p)$, which attains its maximum at $p=\frac{1}{2}\overline{p}_t$. Therefore, the maximum response of incentives to a *unit of shock is given by*

$$\frac{a\sigma\left(T+\sigma^2/h_0\right)^2\left(\overline{p}_t\right)^2}{4\left(T-t\right)^2}=a\sigma\frac{\left(T+\sigma^2/h_0\right)^2\left(\overline{p}^o_t\right)^2}{4\left(T-t\right)^2}+O\left(a^2\right)=\frac{a\sigma\left(T+\sigma^2/h_0\right)^2}{4\left(t+\sigma^2/h_0\right)^2}+O\left(a^2\right).$$

Because the maximum response increases with h_0 and decreases with t according to the above expression, our model has the following empirical predictions.

Empirical Predictions: *Everything else equal, firms which are younger or have a greater amount of profitability uncertainty or human capital risk tend to use more options to compensate their managers.*

Remark 4 *The intuition behind the option-like feature of the optimal contract is as follows. As discussed before, per unit of downward bias in the principal's belief at time t gives rise to a future benefit of increase compensation at time $s \geq t$ by $\frac{h_s\beta_s}{\sigma^2}$. In the benchmark case with deterministic $\{\beta_t\}$, such extra compensation at future date s is uncorrelated with the agent's marginal utility at that time.*

However, in the optimal contract, the principal deliberately makes $\{\beta_t\}$ stochastic. As a result, the future benefit from shirking today will be correlated with the agent's future marginal utility. Moreover, the principal designs the contract optimally so that the correlation will be negative. In this way, she can discourage the agent from cheating early on, because the future benefits are larger usually at times when the agent's marginal utilities are low and thus do not seem as valuable as in the benchmark case.

Lastly, the instantaneous correlation between $\frac{h_s\beta_s}{\sigma^2}$ and $M_s \equiv \exp\left(-\int_0^s a\sigma\beta_v dB_v - \frac{1}{2}\int_0^s a^2\sigma^2\beta_v^2 dv\right)$ is indeed non greater than zero, which is equal to $\left(\frac{h_s}{\sigma^2}\frac{T+\sigma^2/h_0}{T-s}\sigma^p_s\right)\left(-a\sigma\beta_s\right) = -\,a\sigma\frac{\left(T+\sigma^2/h_0\right)\beta_s\sigma^p_s}{(T-s)(s+\sigma^2/h_0)} \leq 0.$

4.3 Numerical Algorithm

The asymptotic analysis in the previous subsection gives very tractable (approximate) characterization of the optimal contract when the agent is sufficiently risk tolerant (i.e., when a is very small). It facilitates the illustration of the option-like feature of the optimal contract and the underlying intuition. The asymptotic analysis results are however silent about the characterization of the optimal contract when the agent is very risk averse (say, $a > 1$). In this section, we propose an innovative numerical algorithm to solve the general model including the cases where the agent is very risk averse. We also report several interesting comparative statics results in this section, which further help us to gain more insights about the optimal contract.

The next proposition plays a very important role in our numerical algorithm. In this proposition below, we derive an important property about the value function $V(t, \overline{p}_t)$ along the upper boundary, which can be fully determined by the upper boundary $\overline{p}_t$ as shown below.

Proposition 6 *Along the upper boundary, the total differentiation of the value function $V(t, \overline{p}_t)$ with respect to time t is given by*

$$\frac{dV(t, \overline{p}_t)}{dt} = \frac{1}{2}\left[\frac{a\sigma^2}{1+a\sigma^2}(1+\overline{p}_t)^2 - 1\right]. \tag{52}$$

Given the terminal condition that $\overline{p}_T = 0$ and $V(T, \overline{p}_T) = 0$, it follows that

$$V(t, \overline{p}_t) = -\int_t^T \left[\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}(1+\overline{p}_s)^2 - \frac{1}{2}\right] ds. \tag{53}$$

The new boundary condition in Eq. (53) is induced by the condition that $\partial V(t, \overline{p}_t)/\partial p = 0$. One can show that only on the true optimal upper boundary can these two conditions hold simultaneously, which is the foundation for the numerical algorithm we propose below. The key in the proposed numerical algorithm is the boundary updating procedure. The idea is that we start with a very loose upper boundary (i.e., a larger domain below it). If this initial choice of the upper boundary happens to be optimal, the principal's value function should attain its maximum along it; otherwise, the maximum should obtain from some interior points provided that the initial upper boundary is chosen to be high enough.

The following lemma is helpful in that it shows we can choose $\{B_t/A_t\}$ as our initial guess of the optimal boundary. Note that in the case of a risk-neutral agent, as we shown in the previous subsection, $\{B_t/A_t\}$ is indeed the optimal upper boundary. Moreover, the lemma shows that if we choose $\{B_t/A_t\}$ as the initial guess for the upper boundary, then the right-hand-size integral in Eq. (53) will exactly equal the value function $V^*(t, B_t/A_t)$ in the benchmark case with deterministic policies. This result is not surprising. In fact, it confirms the optimality of the deterministic policies we obtained in Section 3 where σ^p is constrained to be zero by assumption. Given that by offering the optimal contract, the principal is able to achieve higher value and that the right-hand-side integral in Eq. (53) is greater if the upper boundary is shifted downward, one natural conjecture is that the true upper boundary $\{\overline{p}_t\}$ should not be above $\{B_t/A_t\}$. This conjecture is confirmed by our numerical results.

Lemma 1

$$V^*(t, B_t/A_t) = -\int_t^T \left[\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}\left(1+\frac{B_s}{A_s}\right)^2 - \frac{1}{2}\right] ds. \tag{54}$$

This lemma implies that for any $t \in [0,T)$[17]

$$\int_t^T (1+\overline{p}_s)^2\, ds \le \int_t^T (1+B_s/A_s)^2\, ds. \tag{55}$$

Based on Propositions 4 and 6 and the lemma above, we propose the following numerical algorithm to solve for the optimal contract in the general case. We first start with the initial guess of the upper boundary $\overline{p}_t^{(0)} = \widehat{p}_t$, and then from Proposition 6, we can determine the value function $V^{(0)}\left(t, \overline{p}_t^{(0)}\right)$ along this initial upper boundary. With Dirichlet boundary conditions $V\left(t, \overline{p}_t^{(0)}\right) = V^{(0)}\left(t, \overline{p}_t^{(0)}\right)$ along the boundary $\overline{p}_t^{(0)}$ and $V(t,0) = 0$ along the lower boundary $p = 0$, we can solve the HJB equation numerically, for example, using a finite-difference scheme. Denote the solution by $V^{(1)}(t, p_t)$ for $(t, p_t) \in \Omega^{(0)} \equiv [0,T] \times \left[0, \overline{p}_t^{(0)}\right]$.

Next, we update the upper boundary to $\overline{p}_t^{(1)}$, which is the key element of our numerical algorithm. Fix any time t, we focus on the value function $V^{(1)}(t,p)$ for $p \in \left[0, \overline{p}_t^{(0)}\right]$. Had the boundary $\overline{p}_t^{(0)}$ been optimal, the maximum of $V^{(1)}(t,p)$ should attain at $\overline{p}_t^{(0)}$. Otherwise, the maximum should attain at some interior point, denoted by $\overline{p}_t^{(1)}$.

We repeat the above steps until either $\overline{p}_t^{(n)}$ or $V^{(n)}$ converges. Upon convergence, on the converged upper boundary $\overline{p}_t$, it should satisfy $V_p(t, \overline{p}_t) = 0$ and at the same time satisfy the boundary condition $V(t, \overline{p}_t) = -\int_t^T \left[\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}(1+\overline{p}_s)^2 - \frac{1}{2}\right] ds$.

Remark 5 *Because of singularity at T for the value function $V(t,p)$, we truncate the time interval to $[0, T-\epsilon_T]$ where $\epsilon_T > 0$ is a constant that can be chosen as close as possible to zero. Given the*

[17] Otherwise, suppose there exists $t_1 \in [0,T)$ such that $\int_{t_1}^T (1+\overline{p}_s)^2\, ds > \int_{t_1}^T (1+B_s/A_s)^2\, ds$, then $V\left(t_1, \overline{p}_{t_1}\right) < V^*(t_1, B_{t_1}/A_{t_1}) \le V(t_1, B_{t_1}/A_{t_1})$, which contradicts with the fact that the value function $V(t,\cdot)$ attains its maximum at $\overline{p}_t$.

continuity of the value function, when time t is close enough to T, we can approximate well the value function by the counterpart $V^*(t,p)$ in the benchmark case. As a result, we impose the following terminal condition at $t = T - \epsilon_T$ in our numerical implementation:

$$V(T-\epsilon_T, p) \approx V^*(T-\epsilon_T, p) = -\frac{1}{2}A_{T-\epsilon_T}p^2 + B_{T-\epsilon_T}p \text{ for } p \in \left[0, \overline{p}_{T-\epsilon_T}\right].$$

Remark 6 *Figures 6 and 7 report the comparative statics results with respect to the variance of prior (h_0) and the degree of risk aversion (a), respectively. Figure 6(A) confirms our empirical prediction: the higher h_0, the larger is σ^p. By contrast, the relationship is no longer monotonic in Figure 7(A). It is intuitive if we think about two extreme cases: in one extreme case with $a = 0$, the deterministic policies are indeed optimal and hence $\sigma^p = 0$ in this case, while in the other extreme case with a being very large, it is too costly to incentivize the agent to work and consequently it is optimal to simply recommend zero effort and hence $\sigma^p = 0$ as well. Figures 6(B) and 7(B) are also intuitive in that the principal is worse off if there is more permanent risk (higher h_0) or the agent is more risk averse.*

Remark 7 *Figure 8 plots one simulation path of the equilibrium outcomes. Subplot (1,1) plots the simulated effort path in the general case (blue solid lines) vs in the benchmark case (red dashed lines). Subplot (1,2) plots the simulated changes in incentives (multiplied by 10, blue solid lines) vs the changes in output (green dashed lines). Subplot(2,1) plots the simulated incentive path (i.e. β_t) in the general case (blue solid lines) vs in the benchmark case (red dashed lines). Subplot (2,2) plots the state variable (i.e., p_t or information rent) path in the general case (blue solid lines) vs in the benchmark case (red dashed lines). The parameter values are given by: $a = \sigma = h_0 = k = 1$ and $m_0 = 0$.*

Specifically, Subplot (1,1) shows that in general the optimal contract makes the agent work harder. Moreover, the incentives increase following good output performance as shown in Subplot (1,2). So the simulation results are consistent with the option-like feature of the optimal contract.

5 Conclusion

This paper introduces the agent's ability uncertainty into the Holmstrom and Milgrom (1987) dynamic agency model, and investigates the insurance of permanent risk when the moral hazard issue is present.

Although permanent risk allows both parties to learn from past output history, the agent's dynamic shirking incentives in distorting the principal's belief downward unravels this potential contracting gain to some extent. We fully characterize the optimal contract and find that the optimal contract has some option-like feature, which is used to reduce the information rent which the agent extracts through belief manipulation.

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Appendix

Proof of Proposition 1. Define the agent's expected utility, given that he uses $\mu = \mu^*$ and measured with respect to the principal's information set, as

$$\mathcal{V}_t \equiv \mathbb{E}_t \left[-a \exp \left[-a \left(C_T - G_T \right) \right] | \mathcal{Y}_t, \mu = \mu^* \right].$$

Participation constraint must binds exactly, and hence $\mathcal{V}_0 = \hat{U}$. Because $\mathcal{V}_t$ is a martingale with respect to the information set $\{\mathcal{Y}_t, \mu = \mu^*\}$, by Martingale Representation Theorem, there exists a progressive measurable process $\{\phi_t\}$ such that

$$d\mathcal{V}_t = \phi_t \left(dY_t - \left(\mu_t^* + m_t^* \right) dt \right)$$

Notice that $\mathcal{V}_T = -\exp\left[-a \left(C_T - G_T^* \right) \right]$. Let us define the process C_t so that

$$\mathcal{V}_t = -\exp\left[-a \left(C_t - G_t^* \right) \right]$$

and $C_0 \geq -\frac{1}{a} \ln(-\widehat{U})$ which ensures the agent's participation constraint at time 0. Then C_T must be the time T value of the process C_t. Therefore,

$$d\mathcal{V}_t = \exp\left[-a \left(C_t - G_t^* \right) \right] \left[dC_t - g\left(\mu_t^* \right) dt - \frac{1}{2} a \left(dC_t \right)^2 \right] = \phi_t \left(dY_t - \left(\mu_t^* + m_t^* \right) dt \right).$$

After matching terms, let $\beta_t = \phi_t \exp\left[a \left(C_t - G_t^* \right) \right]$, then

$$dC_t = g\left(\mu_t^* \right) dt + \frac{1}{2} a \beta_t^2 \sigma^2 dt + \beta_t \left(dY_t - \left(\mu_t^* + m_t^* \right) dt \right).$$

■

Proof of Proposition 2. We now proceed to show the link between the incentive β_t and the optimal effort level μ_t^*. Given the contact dynamics:

$$dC_t = g\left(\mu_t^* \right) dt + \frac{1}{2} a \beta_t^2 \sigma^2 dt + \beta_t \left(dY_t - \left(\mu_t^* + m_t^* \right) dt \right).$$

Conditional on the path of effort μ^t, the agent forms his posterior belief, $m_t^\mu = E\left[\theta \left| \mathcal{B}_t \right. \right]$, and $h_t = E\left[\left(\theta - m_t^\mu \right)^2 \left| \mathcal{B}_t \right. \right]$, which evolve as follows,

$$dm_t^\mu = h_t \frac{dY_t - \left(\mu_t + m_t^\mu \right) dt}{\sigma^2} \equiv \frac{h_t}{\sigma} dB_t^\mu, \text{ and } h_t \equiv \frac{\sigma^2 h_0}{\sigma^2 + h_0 t}$$

Therefore, based on the agent's information set, the dynamics of C_t are given by

$$dC_t = g\left(\mu_t^* \right) dt + \frac{1}{2} a \beta_t^2 \sigma^2 dt + \beta_t \left(\mu_t - \mu_t^* + \Delta_t \right) dt + \beta_t \sigma dB_t^\mu.$$

We want to show that $\mu_t^* = \frac{1}{k} \left(\beta_t - \int_t^T \frac{h_s}{\sigma^2} \beta_s ds \right) \equiv \frac{1}{k} \left(\beta_t - \eta_t \right)$ is incentive compatible. To this end, we only need to show that $\mu_t^* = \frac{1}{k} \left(\beta_t - \eta_t \right)$ is the solution of the agent's following optimization problem:

$$\widehat{\mathcal{V}}_t \equiv \max_\mu \mathbb{E}_t^A \left[-\exp\left[-a \left(C_T - G_T \right) \right] | \mathcal{Y}_t, \mu \right].$$

Notice that

$$d\Delta_t = \frac{h_t}{\sigma^2}\left(\mu_t^* - \mu_t - \Delta_t\right) dt. \tag{56}$$

We conjecture that the agent's value function takes the form of

$$\widehat{\mathcal{V}}_t = V\left(t, C_t, G_t, \Delta_t\right) = -\exp\left(-a\left(C_t - G_t\right)\right) F\left(t, \Delta_t\right).$$

Notice that

$$d\Delta_t = \frac{\gamma_t}{\sigma^2}\left(\mu_t^* - \mu_t - \Delta_t\right) dt.$$

The agent's HJB equation can be written as

$$\begin{aligned} 0 &= \max_{\mu_t}\left[\begin{array}{c} V_t + V_G g\left(\mu_t\right) + V_C\left(\beta_t\left(\mu_t - \mu_t^* + \Delta_t\right) + \frac{1}{2}a\beta_t^2\sigma^2 + g\left(\mu_t^*\right)\right) \\ +\frac{1}{2}V_{CC}\left(\beta_t\sigma\right)^2 + V_\Delta\frac{\gamma_t}{\sigma^2}\left(\mu_t^* - \mu_t - \Delta_t\right) \end{array}\right] \\ &= \max_{\mu_t} V\left[\begin{array}{c} \frac{F_t}{F} + ag\left(\mu_t\right) + \frac{1}{2}a^2\left(\beta_t\sigma\right)^2 + \frac{F_\Delta}{F}\frac{\gamma_t}{\sigma^2}\left(\mu_t^* - \mu_t - \Delta_t\right) \\ -a\left(\beta_t\left(\mu_t - \mu_t^* + \Delta_t\right) + \frac{1}{2}a\beta_t^2\sigma^2 + g\left(\mu_t^*\right)\right) \end{array}\right] \end{aligned}$$

Hence, we further conjecture

$$F\left(t, \Delta_t\right) = \exp\left(f\left(t\right)\Delta_t\beta_t\right),$$

then we have

$$0 = \max_{\mu_t}\left[\begin{array}{c} \Delta_t f'\left(t\right) + ag\left(\mu_t\right) + \frac{1}{2}a^2\left(\beta_t\sigma\right)^2 + f\left(t\right)\frac{\gamma_t}{\sigma^2}\left(\mu_t^* - \mu_t - \Delta_t\right) \\ -a\left(\beta_t\left(\mu_t - \mu_t^* + \Delta_t\right) + \frac{1}{2}a\beta_t^2\sigma^2 + g\left(\mu_t^*\right)\right) \end{array}\right]$$

As a result, the first order condition implies that the optimal effort $\hat{\mu}_t$ is given by

$$\hat{\mu}_t = f\left(t\right)\frac{\gamma_t}{ak\sigma^2} + \frac{\beta_t}{k}$$

and

$$f'\left(t\right) = f\left(t\right)\frac{\gamma_t}{\sigma^2} + a\beta_t$$

with the boundary condition $f\left(T\right) = 0$. The solution to the above ODE is given by

$$f\left(t\right) = -a\left(t + \sigma^2/\gamma_0\right)\int_t^T \frac{\beta_s}{s + \sigma^2/\gamma_0} ds. \tag{57}$$

Therefore, we have the expression for the optimal effort level,

$$\hat{\mu}_t = \frac{1}{k}\left[\beta_t - \int_t^T \frac{\beta_s}{s + \sigma^2/\gamma_0} ds\right] \equiv \frac{1}{k}\left(\beta_t - \eta_t\right) = \mu_t^*, \tag{58}$$

where we denote

$$\eta_t \equiv \int_t^T \frac{\beta_s}{s + \sigma^2/\gamma_0} ds \tag{59}$$

Hence, this optimal effort is exactly the same with our conjectured optimal effort level. Finally, by a standard verification theorem, we know that $\mu_t^* = \frac{1}{k}\left(\beta_t - \eta_t\right)$ is indeed incentive compatible. We therefore complete the proof for this proposition. ■

Proof of Theorem 1. The principal's maximization problem is:

$$\max_{\{\beta_t\}} \int_0^T \mu_t dt - \int_0^T \frac{k\mu_t^2}{2} dt - \int_0^T \frac{a\beta_t^2\sigma^2}{2} dt$$
$$s.t. \quad \mu_t = \frac{1}{k}\left(\beta_t - \int_t^T \frac{h_s}{\sigma^2}\beta_s ds\right)$$

Denote $p_t \equiv \int_t^T \frac{h_s}{\sigma^2}\beta_s ds$, then

$$\begin{aligned}\mu_t &= \frac{1}{k}(\beta_t - p_t) \\ p_t' &= -\frac{h_t}{\sigma^2}\beta_t\end{aligned}$$

Therefore, the objective function can be expressed as

$$\begin{aligned}&\max_{\{p_t\}} \int_0^T \left(\frac{1}{k}(\beta_t - p_t) - \frac{1}{2k}(\beta_t - p_t)^2 - \frac{a\sigma^2}{2}\beta_t^2\right) \\ =& \max_{\{p_t\}} \int_0^T \left(\frac{1}{k}\left(-\frac{\sigma^2}{h_t}p_t' - p_t\right) - \frac{1}{2k}\left(-\frac{\sigma^2}{h_t}p_t' - p_t\right)^2 - \frac{a\sigma^2}{2}\left(-\frac{\sigma^2}{h_t}p_t'\right)^2\right) \\ \equiv& \max_{\{p_t\}} \int_0^T L(t, p_t, p_t')\, dt\end{aligned}$$

where $L(t, p_t, p_t') \equiv -\frac{1}{k}\left(p_t + \frac{\sigma^2}{h_t}p_t'\right) - \frac{1}{2k}\left(p_t + \frac{\sigma^2}{h_t}p_t'\right)^2 - \frac{a\sigma^2}{2}\left(\frac{\sigma^2}{h_t}p_t'\right)^2$. This is a standard problem of calculus of variation.

Here we prove a more general problem as follows: starting from time t and arbitrary p_t, we choose the optimal path $\{p_s\}_{s\geq t}$ to solve the following maximization problem

$$V^*(t, p_t) \equiv \max_{\{p_s\}} \int_t^T L(s, p_s, p_s')\, ds$$

where $V^*(t, p_t)$ denotes the optimal value function. The associated Euler equation is

$$L_p(s, p_s, p_s') = \frac{dL_{p'}(s, p_s, p_s')}{ds} \tag{60}$$

Note that

$$\begin{aligned}L_p(s, p_s, p_s') &= -\frac{1}{k} - \frac{1}{k}\left(p_s + \frac{\sigma^2}{h_s}p_s'\right) \\ L_{p'}(s, p_s, p_s') &= -\frac{\sigma^2}{kh_s} - \frac{\sigma^2}{kh_s}\left(p_s + \frac{\sigma^2}{h_s}p_s'\right) - a\sigma^2\left(\frac{\sigma^2}{h_s}\right)^2 p_s'\end{aligned}$$

Also note that $\frac{d(\sigma^2/h_s)}{ds} = 1$, from Eq. (60) it follows that

$$\begin{aligned}&-\frac{1}{k} - \frac{1}{k}\left(p_s + \frac{\sigma^2}{h_s}p_s'\right) \\ =& -\frac{1}{k} - \frac{1}{k}\left(p_s + \frac{\sigma^2}{h_s}p_s'\right) - \frac{\sigma^2}{kh_s}\left(2p_s' + \frac{\sigma^2}{h_s}p_s''\right) - 2a\sigma^2\left(\frac{\sigma^2}{h_s}\right)p_s' - a\sigma^2\left(\frac{\sigma^2}{h_s}\right)^2 p_s''\end{aligned}$$

or

$$\left(1/k + a\sigma^2\right)\left(\frac{\sigma^2}{h_s}\right)^2 p_s'' = -2\left(1/k + a\sigma^2\right)\left(\frac{\sigma^2}{h_s}\right)p_s'$$

or

$$p_s''\left(\frac{\sigma^2}{h_0} + s\right) = -2p_s'$$

The general solution of this ODE is

$$p_s = C_1\left(\frac{\sigma^2}{h_0} + s\right)^{-1} + C_2$$

We choose C_1 and C_2 to satisfy the initial condition p_t and the terminal condition $p_T = 0$. It implies

$$C_1 = -\frac{p_t}{\left(\frac{\sigma^2}{h_0} + T\right)^{-1} - \left(\frac{\sigma^2}{h_0} + t\right)^{-1}} \text{ and } C_2 = \frac{p_t\left(\frac{\sigma^2}{h_0} + T\right)^{-1}}{\left(\frac{\sigma^2}{h_0} + T\right)^{-1} - \left(\frac{\sigma^2}{h_0} + t\right)^{-1}}$$

Therefore,

$$p_s = \frac{\left(\frac{\sigma^2}{h_0} + s\right)^{-1} - \left(\frac{\sigma^2}{h_0} + T\right)^{-1}}{\left(\frac{\sigma^2}{h_0} + t\right)^{-1} - \left(\frac{\sigma^2}{h_0} + T\right)^{-1}} p_t = \frac{\frac{h_s}{h_T} - 1}{\frac{h_t}{h_T} - 1} p_t \tag{61}$$

Because $dp_s = -p_s^2/\sigma^2 ds$, it follows

$$p_s' = -\frac{h_s^2/\left(h_T\sigma^2\right)}{h_t/h_T - 1} p_t \tag{62}$$

and $\frac{\sigma^2 p_s'}{h_s} + p_s = -\frac{1}{h_t/h_T - 1} p_t$. Thus

$$\begin{aligned} L\left(s, p_s, p_s'\right) &= -\frac{1}{k}\left(\frac{\sigma^2 p_s'}{h_s} + p_s\right) - \frac{\left(\frac{\sigma^2 p_s'}{h_s} + p_s\right)^2}{2k} - \frac{a\sigma^2}{2}\left(\frac{\sigma^2 p_s'}{h_s}\right)^2 \\ &= \frac{1}{k}\frac{1}{h_t/h_T - 1} p_t - \frac{1}{2k}\left(\frac{1}{h_t/h_T - 1}\right)^2 p_t^2 - \frac{a\sigma^2}{2}\left(\frac{h_s/h_T}{h_t/h_T - 1}\right)^2 p_t^2 \end{aligned}$$

and

$$\begin{aligned} & V^*\left(t, p_t\right) \\ = & \int_t^T L\left(p_s, p_s', s\right) ds \\ = & \frac{1}{k}\left(\int_t^T \frac{1}{h_t/h_T - 1} ds\right) p_t - \frac{1}{2}\left[\frac{1}{k}\int_t^T \left(\frac{1}{h_t/h_T - 1}\right)^2 ds + a\sigma^2 \int_t^T \left(\frac{h_s/h_T}{h_t/h_T - 1}\right)^2 ds\right] p_t^2 \\ \equiv & -\frac{1}{2} A_t p_t^2 + B_t p_t \end{aligned}$$

where

$$\begin{aligned}
A_t &\equiv \frac{1}{k}\int_t^T \left(\frac{1}{h_t/h_T-1}\right)^2 ds + a\sigma^2 \int_t^T \left(\frac{h_s/h_T}{h_t/h_T-1}\right)^2 ds = \frac{(T-t)/k + a\sigma^4 (h_t-h_T)/h_T^2}{(h_t/h_T-1)^2} \\
&= \frac{\sigma^4}{(T-t)\,h_t}\left(\frac{1}{kh_t}+\frac{a\sigma^2}{h_T}\right) \\
B_t &\equiv \frac{1}{k}\int_t^T \frac{1}{h_t/h_T-1} ds = \frac{1}{k}\frac{T-t}{h_t-h_T}h_T = \frac{\sigma^2}{kh_t}
\end{aligned}$$

The above analysis is very general in that it characterizes the optimal strategies on a possibly off-equilibrium path. To complete the proof, we now derive the equilibrium outcomes along the equilibrium path. When $t=0$, the principal chooses p_0^* to maximize $V^*(0,p)$.

$$p_0^* = \arg\max_p V^*(0,p) = \frac{B_0}{A_0} = \frac{\sigma^2/(kh_0)}{\frac{\sigma^4}{Th_0}\left(\frac{1}{kh_0}+\frac{a\sigma^2}{h_T}\right)} = \frac{Th_0}{\sigma^2(1+ak\Omega)}$$

Starting from p_0^*, from the above analysis, we have

$$\begin{aligned}
p_t^* &= \frac{h_t/h_T-1}{h_0/h_T-1}p_0^* = \frac{h_t(T-t)}{h_0T}p_0^* = \frac{h_t(T-t)}{\sigma^2(1+ak\Omega)} = \frac{h_t/h_T-1}{1+ak\Omega} \\
\beta_t^* &= -\frac{\sigma^2}{h_t}(p_t^*)' = \frac{\sigma^2}{h_t}\frac{h_t^2/\left(h_T\sigma^2\right)}{1+ak\Omega} = \frac{h_t}{h_T}\frac{1}{1+ak\Omega} = \frac{\sigma^2}{(T-t)\,h_T}p_t^* \\
\mu_t^* &= \frac{1}{k}(\beta_t^*-p_t^*) = \frac{1/k}{1+ak\Omega}
\end{aligned}$$

■

Proof of Proposition 3. Fix an arbitrary contract $\{C_T\}$ and an effort strategy $\mu=\{\mu_t\}$, define the agent's continuation value as

$$\mathcal{U}_t^\mu \equiv \mathbb{E}_t^\mu\left[\underbrace{-\exp\left[-a\left(C_T-G_T\right)\right]}_{\equiv\mathcal{U}_T^\mu}|\mathcal{Y}_t,\mu\right] \equiv \mathbb{E}_t^\mu\left[\mathcal{U}_T^\mu\right].$$

Conditional on the path of effort μ^t, the agent forms his posterior belief, $m_t^\mu = E[\theta\,|\mathcal{B}_t]$, and $h_t = E\left[(\theta-m_t^\mu)^2|\mathcal{B}_t\right]$, which evolve as follows,

$$dm_t^\mu = h_t\frac{dY_t-(\mu_t+m_t^\mu)\,dt}{\sigma^2} \equiv \frac{h_t}{\sigma}dB_t^\mu, \text{ and } h_t \equiv \frac{\sigma^2h_0}{\sigma^2+h_0t}.$$

By Martingale Representation Theorem, there exists a progressive measurable process $\{\phi_t^\mu\}$ such that

$$d\mathcal{U}_t^\mu = \phi_t^\mu\left(dY_t-(\mu_t+m_t^\mu)\,dt\right)$$

Define the agent's certainty-equivalent wealth by C_t^μ such that

$$\mathcal{U}_t^\mu = -\exp\left[-a\left(C_t^\mu-G_t\right)\right]$$

Then similarly as before, if we let $\beta_t^\mu \equiv \phi_t^\mu \exp\left[a\left(C_t^\mu - G_t\right)\right]$, then

$$dC_t^\mu = g\left(\mu_t\right)dt + \frac{1}{2}a\sigma^2\left(\beta_t^\mu\right)^2 dt + \beta_t^\mu\left(dY_t - \left(\mu_t + m_t^\mu\right)dt\right)$$

Consider the Brownian motion $\overline{B} = \left\{\overline{B}_t\right\}$ under some probability space with probability measure $\overline{\mathcal{Q}}$ such that

$$Y_t = \int_0^t \sigma d\overline{B}_s$$

Then the following is a martingale under the measure $\overline{\mathcal{Q}}$,

$$\Lambda_{t,T}^\mu \equiv \exp\left(\int_t^T \left(\frac{m_s^\mu + \mu_s}{\sigma}\right)d\overline{B}_s - \frac{1}{2}\int_t^T \left(\frac{m_s^\mu + \mu_s}{\sigma}\right)^2 ds\right)$$

By Girsanov theorem,

$$B_t^\mu \equiv \overline{B}_t - \int_0^t \left(\frac{m_s^\mu + \mu_s}{\sigma}\right)ds$$

is a Brownian motion under the new probability measure $\mathcal{Q}^\mu$ and $d\mathcal{Q}^\mu/d\overline{\mathcal{Q}} = \Lambda_{0,T}^\mu$. Given that both measures are equivalent, the triple $(Y, B^\mu, \mathcal{Q}^\mu)$ is a weak solution of the SDE

$$Y_t = \int_0^t \left(m_s^\mu + \mu_s\right)ds + \int_0^t \sigma dB_s^\mu$$

Therefore,

$$\mathcal{U}_t^\mu \equiv \mathbb{E}_t^\mu\left[\mathcal{U}_T^\mu\left|\mathcal{Y}_t, \mu\right.\right] = \mathbb{E}_t\left[\Lambda_{t,T}^\mu \mathcal{U}_T^\mu\left|\mathcal{Y}_t, \mu\right.\right]$$

where $\mathbb{E}_t^\mu\left[\cdot\right]$ and $\mathbb{E}_t\left[\cdot\right]$ are conditional expectations under the probability measures $\mathcal{Q}^\mu$ and $\overline{\mathcal{Q}}$, respectively.

We use the variational argument following Cvitanic, Wan, and Zhang (2009) and Prat and Jovanovic (2010), and define the control perturbation

$$\mu^\epsilon \equiv \mu + \epsilon\Delta\mu$$

Note that

$$\nabla\left[\int_0^t \mu_s ds\right] \equiv \lim_{\epsilon\longrightarrow 0}\frac{\left[\int_0^t \mu_s^\epsilon ds\right] - \left[\int_0^t \mu_s ds\right]}{\epsilon} = \int_0^t \Delta\mu_s ds$$

Furthermore, because

$$m_t^{\mu^\epsilon} - m_t^\mu = -\frac{h_t}{\sigma^2}\int_0^t \left(\mu_s^\epsilon - \mu_s\right)ds = -\frac{\epsilon h_t}{\sigma^2}\int_0^t \Delta\mu_s ds$$

$$
\begin{aligned}
&\nabla\Lambda_{t,T}^{\mu} \\
\equiv\ & \lim_{\epsilon\to 0}\frac{\exp\left(\int_t^T \frac{m_s^{\mu^\epsilon}+\mu_s^\epsilon}{\sigma}d\overline{B}_s-\frac{1}{2}\int_t^T\frac{\left(m_s^{\mu^\epsilon}+\mu_s^\epsilon\right)^2}{\sigma^2}ds\right)-\exp\left(\int_t^T\frac{m_s^\mu+\mu_s}{\sigma}d\overline{B}_s-\frac{1}{2}\int_t^T\frac{\left(m_s^\mu+\mu_s\right)^2}{\sigma^2}ds\right)}{\epsilon} \\
=\ & \Lambda_{t,T}^{\mu}\lim_{\epsilon\to 0}\frac{\left[\exp\left(\int_t^T\left(\frac{m_s^{\mu^\epsilon}+\mu_s^\epsilon-\left(m_s^\mu+\mu_s\right)}{\sigma}\right)d\overline{B}_s-\frac{1}{2}\int_t^T\left[\left(\frac{m_s^{\mu^\epsilon}+\mu_s^\epsilon}{\sigma}\right)^2-\left(\frac{m_s^\mu+\mu_s}{\sigma}\right)^2\right]ds\right)\right]-1}{\epsilon} \\
=\ & \Lambda_{t,T}^{\mu}\lim_{\epsilon\to 0}\frac{\left[\exp\left(\begin{array}{c}\int_t^T\frac{\epsilon}{\sigma}\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)d\overline{B}_s \\ -\frac{1}{\sigma^2}\int_t^T\left(m_s^\mu+\mu_s\right)\left[\left(m_s^{\mu^\epsilon}+\mu_s^\epsilon\right)-\left(m_s^\mu+\mu_s\right)\right]ds\end{array}\right)\right]-1}{\epsilon} \\
=\ & \Lambda_{t,T}^{\mu}\lim_{\epsilon\to 0}\frac{\left[\exp\left(\frac{\epsilon}{\sigma}\left[\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)\left(d\overline{B}_s-\frac{\left(m_s^\mu+\mu_s\right)}{\sigma}ds\right)\right]\right)\right]-1}{\epsilon} \\
=\ & \Lambda_{t,T}^{\mu}\lim_{\epsilon\to 0}\frac{\left[\exp\left(\frac{\epsilon}{\sigma}\left[\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^\mu\right]\right)\right]-1}{\epsilon} \\
=\ & \Lambda_{t,T}^{\mu}\left(\frac{1}{\sigma}\right)\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^\mu
\end{aligned}
$$

and

$$
\nabla\mathcal{U}_T^\mu\equiv\lim_{\epsilon\to 0}\frac{\mathcal{U}_T^{\mu^\epsilon}-\mathcal{U}_T^\mu}{\epsilon}=\lim_{\epsilon\to 0}\frac{\left[-\exp\left[-a\left(C_T-G_T^{\mu^\epsilon}\right)\right]\right]-\left[-\exp\left[-a\left(C_T-G_T^\mu\right)\right]\right]}{\epsilon}
$$

$$
=\ \mathcal{U}_T^\mu\lim_{\epsilon\to 0}\frac{\exp\left[a\left(G_T^{\mu^\epsilon}-G_T^\mu\right)\right]-1}{\epsilon}
$$

Because

$$
\exp\left[a\left(G_T^{\mu^\epsilon}-G_T^\mu\right)\right]=\exp\left[a\left(\int_0^T\frac{1}{2}k\epsilon\Delta\mu_s\left(2\mu_s+\epsilon\Delta\mu_s\right)ds\right)\right]
$$

Thus

$$
\nabla\mathcal{U}_T^\mu=\mathcal{U}_T^\mu\lim_{\epsilon\to 0}\frac{\exp\left[a\left(G_T^{\mu^\epsilon}-G_T^\mu\right)\right]-1}{\epsilon}=\mathcal{U}_T^\mu ak\int_0^T\mu_s\Delta\mu_s ds
$$

Now we are in a position to characterize the variations of the agent's objective with respect to ϵ

$$
\frac{\widehat{\mathcal{U}}_t^{\mu^\epsilon}-\widehat{\mathcal{U}}_t^\mu}{\epsilon}=\frac{\mathbb{E}_t\left[\Lambda_{t,T}^{\mu^\epsilon}\mathcal{U}_{t,T}^{\mu^\epsilon}\,|\mathcal{Y}_t,\mu^\epsilon\right]-\mathbb{E}_t\left[\Lambda_{t,T}^{\mu}\mathcal{U}_{t,T}^{\mu}\,|\mathcal{Y}_t,\mu\right]}{\epsilon}
$$

Then

$$
\begin{aligned}
& \nabla\widehat{\mathcal{U}}_t^{\mu} \equiv \lim_{\epsilon\to 0}\frac{\widehat{\mathcal{U}}_t^{\mu^{\epsilon}}-\widehat{\mathcal{U}}_t^{\mu}}{\epsilon} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}_t\left[\Lambda_{t,T}^{\mu^{\epsilon}}\mathcal{U}_T^{\mu^{\epsilon}} - \Lambda_{t,T}^{\mu}\mathcal{U}_T^{\mu}\,|\mathcal{Y}_t,\mu\right] \\
= & \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}_t\left[\left(\Lambda_{t,T}^{\mu^{\epsilon}}-\Lambda_{t,T}^{\mu}\right)\mathcal{U}_T^{\mu^{\epsilon}}+\Lambda_{t,T}^{\mu}\left(\mathcal{U}_T^{\mu^{\epsilon}}-\mathcal{U}_T^{\mu}\right)|\mathcal{Y}_t,\mu\right] \\
= & \mathbb{E}_t\left[\nabla\Lambda_{t,T}^{\mu}\mathcal{U}_T^{\mu}+\Lambda_{t,T}^{\mu}\nabla\mathcal{U}_T^{\mu}\,|\mathcal{Y}_t,\mu\right] = \mathbb{E}_t\left[\begin{array}{c}\Lambda_{t,T}^{\mu}\frac{1}{\sigma}\left(\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^{\mu}\right)\mathcal{U}_T^{\mu} \\ +\Lambda_{t,T}^{\mu}\mathcal{U}_T^{\mu}ak\int_0^T\mu_s\Delta\mu_s ds\,|\mathcal{Y}_t,\mu\end{array}\right] \\
= & \mathbb{E}_t^{\mu}\left[\mathcal{U}_T^{\mu}\left(\frac{1}{\sigma}\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^{\mu}+ak\int_0^T\mu_s\Delta\mu_s ds\right)|\mathcal{Y}_t,\mu\right] \\
= & \mathbb{E}_t^{\mu}\left[\mathcal{U}_T^{\mu}\left(\frac{1}{\sigma}\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^{\mu}+ak\int_0^T\mu_s\Delta\mu_s ds\right)|\mathcal{Y}_t,\mu\right]
\end{aligned}
$$

Note that

$$
\begin{aligned}
dC_t^{\mu} &= g(\mu_t)\,dt+\frac{1}{2}a\sigma^2\left(\beta_t^{\mu}\right)^2dt+\beta_t^{\mu}\left(dY_t-\left(\mu_t+m_t^{\mu}\right)dt\right) \\
d\left(C_t^{\mu}-G_t^{\mu}\right) &= \frac{1}{2}a\sigma^2\left(\beta_t^{\mu}\right)^2dt+\beta_t^{\mu}\left(dY_t-\left(\mu_t+m_t^{\mu}\right)dt\right)
\end{aligned}
$$

then

$$
d\mathcal{U}_{\tau}^{\mu} = -a\sigma\beta_{\tau}^{\mu}\mathcal{U}_{\tau}^{\mu}dB_{\tau}^{\mu} \equiv \sigma\varsigma_{\tau}dB_{\tau}^{\mu}
$$

By Ito's lemma

$$
\begin{aligned}
& d\left[\mathcal{U}_{\tau}^{\mu}\frac{1}{\sigma}\int_t^{\tau}\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^{\mu}\right] \\
= & \left[\frac{1}{\sigma}\int_t^{\tau}\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^{\mu}\right]\sigma\varsigma_{\tau}dB_{\tau}^{\mu}+\mathcal{U}_{\tau}^{\mu}\frac{1}{\sigma}\left(-\frac{h_{\tau}}{\sigma^2}\int_0^{\tau}\Delta\mu_v dv+\Delta\mu_{\tau}\right)dB_{\tau}^{\mu} \\
& +\sigma\varsigma_{\tau}\frac{1}{\sigma}\left(-\frac{h_{\tau}}{\sigma^2}\int_0^{\tau}\Delta\mu_v dv+\Delta\mu_{\tau}\right)d\tau
\end{aligned}
$$

and

$$
d\left[\mathcal{U}_{\tau}^{\mu}\int_t^{\tau}\mu_s\Delta\mu_s ds\right] = \left[\int_t^{\tau}\mu_s\Delta\mu_s ds\right]\sigma\varsigma_{\tau}dB_{\tau}^{\mu}+\mathcal{U}_{\tau}^{\mu}\mu_{\tau}\Delta\mu_{\tau}d\tau
$$

Hence,

$$\begin{aligned}
& \nabla\widehat{\mathcal{U}}_t^\mu \\
= \;& \mathbb{E}_t^\mu\left[\mathcal{U}_T^\mu\left(\frac{1}{\sigma}\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^\mu+ak\int_0^T\mu_s\Delta\mu_s ds\right)|\mathcal{Y}_t,\mu\right] \\
= \;& ak\mathcal{U}_t^\mu\int_0^t\mu_s\Delta\mu_s ds \text{ (because } \mathcal{U}_t^\mu=\mathbb{E}_t^\mu\left[\mathcal{U}_T^\mu\right]\text{)} \\
& +\mathbb{E}_t^\mu\left[\mathcal{U}_T^\mu\left(\frac{1}{\sigma}\int_t^T\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)dB_s^\mu+ak\int_t^T\mu_s\Delta\mu_s ds\right)|\mathcal{Y}_t,\mu\right] \\
= \;& ak\mathcal{U}_t^\mu\int_0^t\mu_s\Delta\mu_s ds \\
& +\mathbb{E}_t^\mu\left[\begin{array}{c}\int_t^T\left[\begin{array}{c}\varsigma_s\int_t^s\left(-\frac{h_\tau}{\sigma^2}\int_0^\tau\Delta\mu_v dv+\Delta\mu_\tau\right)dB_\tau^\mu+\mathcal{U}_s^\mu\frac{1}{\sigma}\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)\\ +\sigma\varsigma_s\int_t^s\mu_\tau\Delta\mu_\tau d\tau\end{array}\right]dB_s^\mu\\ +\int_t^T\left[\varsigma_s\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)+ak\mathcal{U}_s^\mu\mu_s\Delta\mu_s\right]ds\,|\mathcal{Y}_t,\mu\end{array}\right] \\
= \;& ak\mathcal{U}_t^\mu\int_0^t\mu_s\Delta\mu_s ds+\mathbb{E}_t^\mu\left[\int_t^T\left(\varsigma_s\left(-\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dv+\Delta\mu_s\right)+ak\mathcal{U}_s^\mu\mu_s\Delta\mu_s\right)ds\,|\mathcal{Y}_t,\mu\right]
\end{aligned}$$

where the last equality is based on Lemma 7.3 in Cvitanic, Wan, and Zhang (2009). Note that

$$\int_t^T\varsigma_s\frac{h_s}{\sigma^2}\int_0^s\Delta\mu_v dvds=\int_t^T\int_v^T\varsigma_s\frac{h_s}{\sigma^2}\Delta\mu_v dsdv=\int_t^T\left(\int_s^T\varsigma_\tau\frac{h_\tau}{\sigma^2}d\tau\right)\Delta\mu_s ds$$

Thus

$$\nabla\mathcal{U}_t^\mu=ak\mathcal{U}_t^\mu\int_0^t\mu_s\Delta\mu_s ds+\mathbb{E}_t^\mu\left[\int_t^T\left(-\int_s^T\varsigma_\tau\frac{h_\tau}{\sigma^2}d\tau+\varsigma_s+\mathcal{U}_s^\mu\mu_s\right)\Delta\mu_s ds\,|\mathcal{Y}_t,\mu\right]$$

Because $\Delta\mu_s$ was arbitrary and we can ignore $\Delta\mu_s$ $0\leq s<t$ (or treat them as zero), hence

$$\left(\mathbb{E}_t^\mu\left[-\int_t^T\varsigma_\tau\frac{h_\tau}{\sigma^2}d\tau\right]+\varsigma_t+ak\mathcal{U}_t^\mu\mu_t\right)(\mu_t-\mu_t^*)\leq 0$$

Given that $\varsigma_t=-a\beta_t^\mu\mathcal{U}_t^\mu$, the necessary condition for μ^* is

$$\mu_t=-\frac{1}{ak\mathcal{U}_t^\mu}\left(\varsigma_t-\mathbb{E}_t^\mu\left[\int_t^T\varsigma_\tau\frac{h_\tau}{\sigma^2}d\tau\right]\right)=\frac{1}{k}\left(\beta_t-\frac{1}{\mathcal{U}_t^\mu}\mathbb{E}_t^\mu\left[\int_t^T\beta_\tau^\mu\mathcal{U}_\tau^\mu\frac{h_\tau}{\sigma^2}d\tau\right]\right)$$

When β is deterministic, then $\mu_t=\frac{1}{k}\left(\beta_t-\int_t^T\beta_\tau^\mu\frac{h_\tau}{\sigma^2}d\tau\right)$. However, when β is non-deterministic, the problem becomes much more complicated. ■

Proof of Proposition 4. We are guessing there exists an upper boundary $\overline{p}(t)$ such that

$$V_p(t,\overline{p}(t))=0$$

This implies that $\beta(t,\overline{p}(t))=\frac{1+\overline{p}(t)}{1+a\sigma^2}$ and $\sigma^P(t,\overline{p}(t))=0$. Differentiating HJB equation with respect to p, and using envelope theorem, we have

$$\begin{aligned}
0 &= -1+\beta(t,\overline{p}(t))-\overline{p}(t)+V_{pp}\beta(t,\overline{p}(t))\left(\sigma^P a\sigma-\frac{h_t}{\sigma^2}\right)+\frac{1}{2}V_{ppp}\left(\sigma^P\right)^2+V_{tp} \\
&= -1+\beta(t,\overline{p}(t))-\overline{p}(t)-V_{pp}\beta(t,\overline{p}(t))\frac{h_t}{\sigma^2}+V_{tp} \\
&= -a\sigma^2\frac{1+\overline{p}(t)}{1+a\sigma^2}-V_{pp}\beta(t,\overline{p}(t))\frac{h_t}{\sigma^2}+V_{tp}
\end{aligned}$$

suppose that $V_{pp} < 0$. Then

$$\frac{V_{tp}\left(t,\overline{p}\left(t\right)\right)}{-V_{pp}\left(t,\overline{p}\left(t\right)\right)}-\left[-\beta\left(t,\overline{p}\left(t\right)\right)\frac{h_t}{\sigma^2}\right]=\frac{a\sigma^2\frac{1+\overline{p}(t)}{1+a\sigma^2}}{-V_{pp}}>0$$

therefore the slope of the path $\overline{p}\left(t\right)$ satisfies

$$\overline{p}'\left(t\right)=\frac{V_{tp}\left(t,\overline{p}\left(t\right)\right)}{-V_{pp}\left(t,\overline{p}\left(t\right)\right)}>-\beta\left(t,\overline{p}\left(t\right)\right)\frac{h_t}{\sigma^2}=\left.\frac{dp}{dt}\right|_{p=\overline{p}(t)}$$

By a similar argument, we prove below that $p=0$ is absorbing. If there exists a lower boundary $\underline{p}\left(t\right)$ such that

$$V_p\left(t,\underline{p}\left(t\right)\right)=\left(1+\underline{p}\left(t\right)\right)\frac{\sigma^2}{h_t} \tag{63}$$

This implies that

$$\beta\left(t,\underline{p}\left(t\right)\right)=\sigma^P\left(t,\underline{p}\left(t\right)\right)=0$$

Differentiating HJB equation with respect to p, and using envelope theorem, we have

$$\begin{aligned} 0 &= -1+\beta\left(t,\underline{p}\left(t\right)\right)-\underline{p}\left(t\right)+V_{pp}\beta\left(t,\underline{p}\left(t\right)\right)\left(\sigma^P a\sigma-\frac{h_t}{\sigma^2}\right)+\frac{1}{2}V_{ppp}\left(\sigma^P\right)^2+V_{tp} \\ &= -1-\overline{p}\left(t\right)+V_{tp} \end{aligned} \tag{64}$$

On the other hand, total differentiation of Eq. (63) with respect to t yields

$$V_{tp}\left(t,\underline{p}\left(t\right)\right)+V_{pp}\left(t,\underline{p}\left(t\right)\right)\underline{p}'\left(t\right)=\left(1+\underline{p}\left(t\right)\right)+\underline{p}'\left(t\right)\frac{\sigma^2}{h_t}$$

Because $V_{tp}\left(t,\underline{p}\left(t\right)\right)=1+\underline{p}\left(t\right)$ (Eq. 64) and $V_{pp}\left(t,\underline{p}\left(t\right)\right)-\frac{\sigma^2}{h_t}<0$, thus it must be true that

$$\underline{p}'\left(t\right)=0$$

This suggests that the lower boundary is independent of time t. Let us denote it by $\underline{p}$, i.e., $\underline{p}\left(t\right)\equiv\underline{p}$. More importantly, it is an absorbing boundary, because

$$\underline{p}'\left(t\right)=0=-\beta\left(t,\underline{p}\left(t\right)\right)\frac{h_t}{\sigma^2}=\left.\frac{dp}{dt}\right|_{p=\underline{p}(t)}$$

We now turn to prove $\underline{p}=0$. The lower boundary must be zero for the following reasons. Firstly, once the lower boundary is hit, say at time t, the state variable p_s remains to be $\underline{p}$ since it is an absorbing boundary, and it is optimal to follow the policies in the benchmark case. From Eq. (38), $p_s=\frac{h_s/h_T-1}{h_t/h_T-1}\underline{p}$. Therefore, in order for p_s to stay constant, it must be true that $\underline{p}=0$. ■

Before we prove Proposition 6, we need to prove the following lemma first, which is used in the proof later.

Lemma 2 $\overline{p}_T=0$

Proof of Lemma 2. We prove it by contradiction. If $\overline{p}_T$ is not equal to zero, there are only two possibilities: (i) $\overline{p}_T > 0$ or (ii) $\overline{p}_{T_0} = 0$ and $T_0 < T$ where T_0 is the earliest time when $\overline{p}_t$ hits zero.

(i) is impossible because there is always a positive probability that $p_T > 0$, violating the boundary condition $p_T = 0$.

Next, we prove (ii). Because $T_0 < T$ and $B_{T_0}/A_{T_0} > 0$. At time T_0, for any $0 < p < B_{T_0}/A_{T_0}$, we have $V^*(T_0, p) = -\frac{1}{2}A_{T_0}p^2 + B_{T_0}p > 0$ and $V_p^*(T_0, p) = -A_{T_0}p + B_{T_0} > 0$. Thus

$$\begin{aligned} 0 &= V_p(T_0, p_{T_0}) = V_p(T_0, 0) \\ &= \lim_{p\to 0} \frac{V(T_0, p) - V(T_0, 0)}{p} = \lim_{p\to 0} \frac{V(T_0, p)}{p} \\ &\geq \lim_{p\to 0} \frac{V^*(T_0, p)}{p} = \lim_{p\to 0} \frac{V^*(T_0, p) - V^*(T_0, 0)}{p} = V_p^*(T_0, 0) = B_{T_0} > 0 \end{aligned}$$

■

Proof of Proposition 5. In the asymptotic analysis, we consider the following approximations up to the second order of magnitude of a. That is,

$$\overline{p}_t = \overline{p}_t^o - aq_1(t) - a^2 q_2(t) + O\left(a^3\right), \tag{65}$$

$$V(t, p) = V^o(t, p) + aH_1(t, p) + a^2 H_2(t, p) + O\left(a^3\right). \tag{66}$$

First, we expand the value function $V(t, p)$ and its partial derivatives $V_p(t, p)$ and $V_{pp}(t, p)$ in the HJB equation up to second order of magnitude of a. Note that $\left(1 + af_1 + a^2 f_2\right)^{-1} = 1 - af_1 - a^2 f_2 + a^2 f_1^2 + O\left(a^3\right)$, we have

$$\begin{aligned} 0 &= V_t + \frac{1}{2}\frac{\left(1 + p - \frac{h_t}{\sigma^2}V_p\right)^2}{1 + a\sigma^2 + a^2\sigma^2\frac{V_p^2}{V_{pp}}} - \frac{1}{2}p^2 - p \\ &= V_t + \frac{1}{2}\left(1 + p - \frac{h_t}{\sigma^2}V_p\right)^2\left(1 - a\sigma^2 - a^2\sigma^2\frac{(V_p)^2}{V_{pp}} + \sigma^4 a^2\right) - \frac{1}{2}p^2 - p + O\left(a^3\right) \\ &= V_t^o + \frac{1}{2}\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)^2 - \frac{1}{2}p^2 - p \\ &\quad + a\left[\frac{\partial H_1}{\partial t} - \frac{h_t}{\sigma^2}\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)\frac{\partial H_1}{\partial p} - \frac{1}{2}\sigma^2\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)^2\right] \\ &\quad + a^2\left[\begin{array}{c} \frac{\partial H_2}{\partial t} + \frac{1}{2}\left(\frac{h_t}{\sigma^2}\right)^2\left(\frac{\partial H_1}{\partial p}\right)^2 - \left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)\frac{h_t}{\sigma^2}\frac{\partial H_2}{\partial p} \\ +\sigma^2\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)\frac{h_t}{\sigma^2}\frac{\partial H_1}{\partial p} + \left(\frac{1}{2}\right)\left(\sigma^4 - \sigma^2\frac{\left(V_p^o\right)^2}{V_{pp}^o}\right)\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)^2 \end{array}\right] + O\left(a^3\right). \end{aligned}$$

Note that $V_t^o + \frac{1}{2}\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)^2 - \frac{1}{2}p^2 - p = 0$ by definition of $V^o(t, p)$. By equating the coefficients of a and a^2 to zero, we obtain

$$\frac{\partial H_1}{\partial t} = \frac{h_t}{\sigma^2}\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)\frac{\partial H_1}{\partial p} + \frac{1}{2}\sigma^2\left(1 + p - \frac{h_t}{\sigma^2}V_p^o\right)^2, \tag{67}$$

and

$$\frac{\partial H_2}{\partial t} = \left(1+p-\frac{h_t}{\sigma^2}V_p^o\right)\frac{h_t}{\sigma^2}\frac{\partial H_2}{\partial p}-\frac{1}{2}\left(\frac{h_t}{\sigma^2}\right)^2\left(\frac{\partial H_1}{\partial p}\right)^2 \tag{68}$$

$$-\sigma^2\left(1+p-\frac{h_t}{\sigma^2}V_p^o\right)\frac{h_t}{\sigma^2}\frac{\partial H_1}{\partial p}-\left(\frac{1}{2}\right)\left(\sigma^4-\sigma^2\frac{\left(V_p^o\right)^2}{V_{pp}^o}\right)\left(1+p-\frac{h_t}{\sigma^2}V_p^o\right)^2.$$

Next, we identify the boundary conditions for $H_1(t,p)$ and $H_2(t,p)$ on the boundaries $p=0$ or $\overline{p}_t$. To satisfy the boundary conditions $V(t,0)=0$ and $V_p(t,0)=\frac{\sigma^2}{h_t}=B_t^o$ along the lower boundary $p=0$, it must hold that

$$H_1(t,0) = \frac{\partial H_1(t,0)}{\partial p}=0, \tag{69}$$

$$H_2(t,0) = \frac{\partial H_2(t,0)}{\partial p}=0, \tag{70}$$

where the second equalities from matching the coefficients of a and a^2 in the following boundary condition

$$\begin{aligned}\frac{\sigma^2}{h_t} &= V_p(t,0)=V_p^o(t,0)-a\frac{\partial H_1(t,0)}{\partial p}-a^2\frac{\partial H_2(t,0)}{\partial p}+O\left(a^3\right)\\ &= \frac{\sigma^2}{h_t}-a\frac{\partial H_1(t,0)}{\partial p}-a^2\frac{\partial H_2(t,0)}{\partial p}+O\left(a^3\right).\end{aligned}$$

We now look at the boundary conditions at the upper boundary $p=\overline{p}_t$. Firstly,

$$\begin{aligned}&V(t,\overline{p}_t)=-\int_t^T\left[\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}(1+\overline{p}_s)^2-\frac{1}{2}\right]ds\\ =\ &V^o(t,\overline{p}_t^o)-\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}\int_t^T(1+\overline{p}_s)^2ds=V^o(t,\overline{p}_t)+\frac{1}{2}A_t^o(\overline{p}_t-\overline{p}_t^o)^2-\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}\int_t^T(1+\overline{p}_s)^2ds\\ =\ &V^o(t,\overline{p}_t)+\frac{1}{2}a^2A_t^o(q_1(t))^2-\frac{1}{2}a\sigma^2\left(1-a\sigma^2\right)\int_t^T(1+\overline{p}_s^o-aq_1(s))^2ds+O\left(a^3\right)\\ =\ &V^o(t,\overline{p}_t)-a\left[\frac{\sigma^2}{2}\int_t^T(1+\overline{p}_s^o)^2ds\right]\\ &+a^2\left[\frac{1}{2}A_t^o(q_1(t))^2+\frac{\sigma^4}{2}\int_t^T(1+\overline{p}_s^o)^2ds+\sigma^2\int_t^Tq_1(s)(1+\overline{p}_s^o)ds\right]+O\left(a^3\right)\end{aligned}$$

On the other hand, we can express $V(t,\overline{p}_t)$ as follows

$$\begin{aligned}V(t,\overline{p}_t) &= V^o(t,\overline{p}_t)+aH_1(t,\overline{p}_t)+a^2H_2(t,\overline{p}_t)+O\left(a^3\right)\\ &= V^o(t,\overline{p}_t)+a\left[H_1(t,\overline{p}_t^o)+\frac{\partial H_1(t,\overline{p}_t^o)}{\partial p}(\overline{p}_t-\overline{p}_t^o)\right]+a^2H_2(t,\overline{p}_t^o)+O\left(a^3\right)\\ &= V^o(t,\overline{p}_t)+aH_1(t,\overline{p}_t^o)+a^2\left[H_2(t,\overline{p}_t^o)-\frac{\partial H_1(t,\overline{p}_t^o)}{\partial p}q_t\right]+O\left(a^3\right).\end{aligned}$$

By matching the coefficients of a and a^2 in the above two equations, we have

$$H_1(t,\overline{p}_t^o) = -\frac{\sigma^2}{2}\int_t^T (1+\overline{p}_s^o)^2\,ds, \tag{71}$$

$$H_2(t,\overline{p}_t^o) = \frac{\partial H_1(t,\overline{p}_t^o)}{\partial p}q_1(t) + \frac{A_t^o}{2}q_1(t)^2 + \frac{\sigma^4}{2}\int_t^T (1+\overline{p}_s^o)^2\,ds + \sigma^2\int_t^T q_1(s)(1+\overline{p}_s^o)\,ds \tag{72}$$

Secondly, we use the condition $V_p(t,\overline{p}_t) = 0$ and also the second-order approximations of $V(t,p)$ and $\overline{p}_t$. It yields

$$\begin{aligned} 0 &= V_p(t,\overline{p}_t) = V_p^o(t,\overline{p}_t) + a\frac{\partial H_1(t,\overline{p}_t)}{\partial p} + a^2\frac{\partial H_2(t,\overline{p}_t)}{\partial p} + O(a^3) \\ &= -A_t^o(\overline{p}_t - \overline{p}_t^o) + a\left(\frac{\partial H_1(t,\overline{p}_t^o)}{\partial p} + \frac{\partial^2 H_1(t,\overline{p}_t^o)}{\partial p^2}(\overline{p}_t - \overline{p}_t^o)\right) + a^2\frac{\partial H_2(t,\overline{p}_t^o)}{\partial p} + O(a^3) \\ &= a\left[\frac{\partial H_1(t,\overline{p}_t^o)}{\partial p} + A_t^o q_1(t)\right] + a^2\left[A_t^o q_2(t) - \frac{\partial^2 H_1(t,\overline{p}_t^o)}{\partial p^2}q_1(t) + \frac{\partial H_2(t,\overline{p}_t^o)}{\partial p}\right] + O(a^3). \end{aligned}$$

It follows from the above equation that

$$A_t^o q_1(t) = -\frac{\partial H_1(t,\overline{p}_t^o)}{\partial p}, \tag{73}$$

$$A_t^o q_2(t) = -\frac{\partial H_2(t,\overline{p}_t^o)}{\partial p} + \frac{\partial^2 H_1(t,\overline{p}_t^o)}{\partial p^2}q_1(t). \tag{74}$$

These two equations will be used to determine $q_1(t)$ and $q_2(t)$ in the boundary approximation once we solve $H_1(t,p)$ and $H_2(t,p)$.

We are now in a position to solve the functions $H_1(t,p)$ and $H_2(t,p)$. Let's solve $H_1(t,p)$ first, which satisfies the following ODE in Eq. (67) along with the boundary conditions in Eqs. (69,71). It is straightforward to show that the ODE has the following general solutions:

$$H_1(t,p) = -C_2\left(\frac{p}{\overline{p}_t^o}\right)^{C_1} - \frac{1}{2}\frac{\sigma^2(T+\sigma^2/h_0)(t+\sigma^2/h_0)}{T-t}p^2$$

where C_1 and $C_2 > 0$ are constants. After tedious algebra, it is easy to check that if $C_2 = 0$, or

$$H_1(t,p) = -\frac{1}{2}\frac{\sigma^2(T+\sigma^2/h_0)(t+\sigma^2/h_0)}{T-t}p^2 \equiv H_1(t)\,p^2 \tag{75}$$

it is the unique solution that satisfies the ODE and the three conditions. From Eq. (73), we have

$$q_1(t) = \frac{\sigma^2(T+\sigma^2/h_0)(T-t)}{(t+\sigma^2/h_0)^2}. \tag{76}$$

Similarly, we can solve for $H_2(t,p)$ and $q_2(t)$. In fact, substituting Eqs. (75) and (76) into Eq. (72), after tedious algebra we can show that Eq. (72) is equivalent to

$$H_2(t,\overline{p}_t^o) = 0.$$

We can also simplify the ODE in Eq. (68) as the following:

$$\frac{\partial H_2(t,p)}{\partial p} = \frac{h_t}{\sigma^2}\frac{T+\sigma^2/h_0}{T-t}pH_p - \frac{1}{2}\frac{\sigma^2\left(T+\sigma^2/h_0\right)^2\left(t+\sigma^2/h_0\right)^2}{(T-t)^3}\left(p-\overline{p}_t^o\right)^2 p^2.$$

To find the general solution of this ODE, we conduct the transformation below:

$$\widehat{p} = \frac{p}{\overline{p}_t^o} \text{ and } H(t,p) = \widehat{H}(t,\widehat{p}).$$

Under the transformation, the ODE of $H_2(t,p)$ becomes

$$\widehat{H}_t = -\frac{1}{2}\frac{\sigma^2\left(T+\sigma^2/h_0\right)^2(T-t)}{\left(t+\sigma^2/h_0\right)^2}\left(\widehat{p}-1\right)^2\widehat{p}^2,$$

which has the general solutions

$$\widehat{H}(t,\widehat{p}) = C_3(t)\left(\widehat{p}-1\right)^2\widehat{p}^2 + C_4(\widehat{p}),$$

where

$$C_3'(t) = -\frac{1}{2}\frac{\sigma^2\left(T+\sigma^2/h_0\right)^3}{\left(t+\sigma^2/h_0\right)^2} + \frac{1}{2}\frac{\sigma^2\left(T+\sigma^2/h_0\right)^2}{\left(t+\sigma^2/h_0\right)},$$

or, under the constraint $C_3(T) = 0$,

$$C_3(t) = \frac{1}{2}\sigma^2\left(T+\sigma^2/h_0\right)^2\left[\frac{(T-t)}{\left(t+\sigma^2/h_0\right)} - \log\left(\frac{T+\sigma^2/h_0}{t+\sigma^2/h_0}\right)\right]. \tag{77}$$

To satisfy the boundary condition $H_2(t,\overline{p}_t^o) = 0$, it must be true that $C_4(\cdot) \equiv 0$. Therefore, $H_2(t,p)$ is given by

$$H_2(t,p) = \frac{C_3(t)}{\left(\overline{p}_t^o\right)^4}\left(p-\overline{p}_t^o\right)^2 p^2 \equiv H_2(t)\left(p-\overline{p}_t^o\right)^2 p^2. \tag{78}$$

Because $\partial H_2(t,\overline{p}_t^o)/\partial p = 0$, the condition in Eq. (74) implies

$$q_2(t) = \frac{q_1(t)}{A_t^o}\frac{\partial^2 H_1(t,\overline{p}_t^o)}{\partial p^2} = -\frac{\sigma^4\left(T+\sigma^2/h_0\right)^2(T-t)}{\left(t+\sigma^2/h_0\right)^3}. \tag{79}$$

Lastly, we derive the second-order approximations for $\beta(t,p)$ and $\sigma^p(t,p)$. We need to analyze $V_p(t,p)$ first, whose approximation is given below

$$\begin{aligned} & V_p(t,p) \\ = & -\left(A_t^o + a\sigma^2\left(T+\sigma^2/h_0\right)\frac{t+\sigma^2/h_0}{T-t}\right)p + B_t^o + 2a^2H_2(t)\left[\left(p-\overline{p}_t^o\right)p^2 + \left(p-\overline{p}_t^o\right)^2 p\right] + O\left(a^3\right) \\ = & -\left(p-\overline{p}_t\right)\left[A_t^o + a\frac{\sigma^2\left(T+\sigma^2/h_0\right)\left(t+\sigma^2/h_0\right)}{(T-t)} - 4a^2H_2(t)\,p\left(p-\frac{1}{2}\overline{p}_t^o\right)\right] + O\left(a^3\right). \end{aligned}$$

Therefore,

$$\begin{aligned}
&\beta(t,p)\\
=\ & \frac{1+p-\frac{h_t}{\sigma^2}V_p}{1+a\sigma^2+a^2\sigma^2\frac{V_p^2}{V_{pp}}} = \left(1+p-\frac{h_t}{\sigma^2}V_p\right)\left(1-a\sigma^2-a^2\sigma^2\frac{\left(V_p^o\right)^2}{V_{pp}^o}+\sigma^4 a^2+O\left(a^3\right)\right)\\
=\ & p\left[\frac{T+\sigma^2/h_0}{T-t}+a^2\underbrace{\left(\frac{\sigma^2\left(T+\sigma^2/h_0\right)\left(t+\sigma^2/h_0\right)^2\left(p-\overline{p}_t^o\right)^2}{\left(T-t\right)^2}-\frac{4H_2(t)\left(p-\overline{p}_t^o\right)\left(p-\frac{\overline{p}_t^o}{2}\right)}{B_t^o}\right)}_{\equiv N_1(t,p)\geq 0}\right]+O\left(a^3\right)\\
\equiv\ & p\left(\frac{T+\sigma^2/h_0}{T-t}+a^2N_1(t,p)\right)+O\left(a^3\right).
\end{aligned}$$

and

$$\begin{aligned}
&\sigma^P(t,p)\\
=\ & -a\sigma\beta\frac{V_p}{V_{pp}} = -a\sigma p\left(\frac{T+\sigma^2/h_0}{T-t}+a^2N_1(t,p)\right)\frac{V_p}{V_{pp}}+O\left(a^3\right)\\
=\ & -a\sigma p\frac{T+\sigma^2/h_0}{T-t}\frac{V_p^o+aG_p}{V_{pp}^o+aG_{pp}}+O\left(a^3\right)\\
=\ & -a\sigma p\frac{T+\sigma^2/h_0}{T-t}\left(\frac{V_p^o}{V_{pp}^o}-\frac{aV_p^o}{\left(V_{pp}^o\right)^2}\frac{\partial^2 H_1(t,p)}{\partial p^2}+\frac{a}{V_{pp}^o}\frac{\partial H_1(t,p)}{\partial p}\right)+O\left(a^3\right)\\
=\ & -a\sigma p\frac{T+\sigma^2/h_0}{T-t}\left(p-\overline{p}_t^o+a\frac{\sigma^2\left(T+\sigma^2/h_0\right)\left(T-t\right)}{\left(t+\sigma^2/h_0\right)^2}\right)+O\left(a^3\right)\\
=\ & -a\sigma p\frac{T+\sigma^2/h_0}{T-t}\left(p-\left(\overline{p}_t^o-aq_t\right)\right)+O\left(a^3\right) = -a\sigma p\frac{T+\sigma^2/h_0}{T-t}\left(p-\overline{p}_t\right)+O\left(a^3\right).
\end{aligned}$$

■

Proof of Proposition 6. Along the upper boundary $\overline{p}_t$, $V_p(t,\overline{p}_t)=0$. Therefore, from the first-order conditions, we have

$$\sigma^P(t,\overline{p}_t)=0 \text{ and } \beta(t,\overline{p}_t)=\frac{1+\overline{p}_t}{1+a\sigma^2}.$$

Furthermore, the total time differentiation of $V(t,\overline{p}_t)$ turns out to be the same as the partial derivative with respect to t, because

$$\frac{dV(t,\overline{p}_t)}{dt}=V_t+V_p(t,\overline{p}_t)\frac{d\overline{p}_t}{dt}=V_t$$

On the other hand, from the HJB equation, we have

$$V_t=\frac{1}{2}\left(\frac{1+\overline{p}_t}{1+a\sigma^2}-\overline{p}_t-1\right)^2-\frac{1}{2}+\frac{a\sigma^2}{2}\left(\frac{1+\overline{p}_t}{1+a\sigma^2}\right)^2=\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}\left(1+\overline{p}_t\right)^2-\frac{1}{2}$$

Note that at T, $\overline{p}_T=0$ from Lemma 2. Because $V(T,0)=0$, we can integrate the above equation to find out

$$V(t,\overline{p}_t)=\int_t^T\left[\frac{1}{2}\frac{a\sigma^2}{1+a\sigma^2}\left(1+\overline{p}_s\right)^2-\frac{1}{2}\right]ds$$

■

Proof of Lemma 1. First, note that $\frac{B_t}{A_t} = \frac{h_t(T-t)}{\sigma^2[1+a\sigma^2(h_t/h_T)]} = \frac{h_t/h_T - 1}{1+a\sigma^2(h_t/h_T)}$. Thus

$$\begin{aligned}\int_t^T \frac{B_s}{A_s} ds &= \int_t^T \frac{h_s/h_T - 1}{1 + a\sigma^2 (h_s/h_T)} ds = \int_t^T \frac{h_0\sigma^2 - h_T (h_0 s + \sigma^2)}{h_T (h_0 s + \sigma^2) + a h_0 \sigma^4} ds \\ &= -(T-t) + \frac{\sigma^2 (1 + a\sigma^2)}{h_T} \log \left(\frac{T + \sigma^2/h_0 + a\sigma^4/h_T}{t + \sigma^2/h_0 + a\sigma^4/h_T} \right),\end{aligned}$$

and

$$\begin{aligned}\int_t^T \left(\frac{B_s}{A_s}\right)^2 ds &= \int_t^T \left[-1 + \frac{\sigma^2 (1 + a\sigma^2)/h_T}{s + \sigma^2/h_0 + a\sigma^4/h_T} \right]^2 ds \\ &= (T-t) - 2\frac{\sigma^2 (1 + a\sigma^2)}{h_T} \log \left(\frac{T + \sigma^2/h_0 + a\sigma^4/h_T}{t + \sigma^2/h_0 + a\sigma^4/h_T} \right) \\ &\quad + \left(\frac{\sigma^2 (1 + a\sigma^2)}{h_T} \right)^2 \left[\frac{1}{t + \sigma^2/h_0 + a\sigma^4/h_T} - \frac{1}{T + \sigma^2/h_0 + a\sigma^4/h_T} \right].\end{aligned}$$

Then

$$\begin{aligned}&\int_t^T \left[\frac{1}{2} \frac{a\sigma^2}{1 + a\sigma^2} \left(1 + \frac{B_s}{A_s} \right)^2 - \frac{1}{2} \right] ds \\ =\ & \frac{1}{2} \left(\frac{a\sigma^2}{1 + a\sigma^2} - 1 \right) (T-t) + \frac{a\sigma^2}{1 + a\sigma^2} \int_t^T \overline{p}_s ds + \frac{1}{2} \frac{a\sigma^2}{1 + a\sigma^2} \int_t^T \overline{p}_s^2 ds \\ =\ & -\frac{T-t}{2} + \frac{a\sigma^2}{2} \frac{\sigma^4 (1 + a\sigma^2)}{h_T^2} \left[\frac{1}{t + \sigma^2/h_0 + a\sigma^4/h_T} - \frac{1}{T + \sigma^2/h_0 + a\sigma^4/h_T} \right].\end{aligned}$$

As a result,

$$\begin{aligned}&-\int_t^T \left[\frac{1}{2} \frac{a\sigma^2}{1 + a\sigma^2} \left(1 + \frac{B_s}{A_s} \right)^2 - \frac{1}{2} \right] ds \\ =\ & \frac{T-t}{2} - \frac{a\sigma^2}{2} \frac{\sigma^4 (1 + a\sigma^2)}{h_T^2} \left[\frac{1}{t + \sigma^2/h_0 + a\sigma^4/h_T} - \frac{1}{T + \sigma^2/h_0 + a\sigma^4/h_T} \right] \\ =\ & \frac{T-t}{2} \left[1 - a\sigma^2 \frac{\sigma^2}{h_T} \frac{1}{(t + \sigma^2/h_0 + a\sigma^4/h_T)} \right] = \frac{T-t}{2} \left[\frac{1}{1 + a\sigma^2 h_t/h_T} \right] = \frac{1}{2} \frac{B_t^2}{A_t} \\ =\ & V^* \left(t, \frac{B_t}{A_t} \right)\end{aligned}$$

■

Figure 2: Value Function $V(t,p)$

This figure plots the optimal upper boundary $\overline{p}_t$ in the case of risk-averse agent $(a = 0.1)$, depicted by blue solid line, as well as the optimal upper boundary $\overline{p}_t^o$ in the case of risk-neutral agent $(a = 0)$. The specification of other parameter values is given by: $T = h_0 = \sigma = k = 1$.

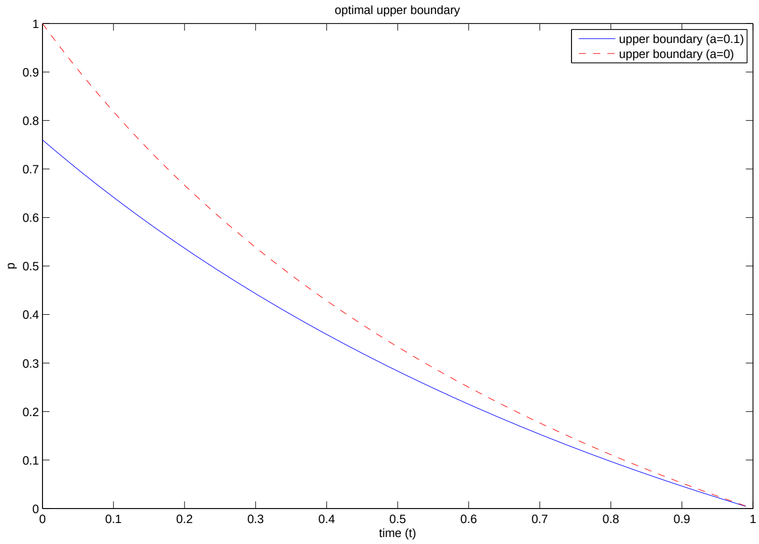

Figure 3: Value Function $V(t,p)$

This figure plots the principal's value function $V(t,p)$ under the optimal contract based on the asymptotic analysis. The parameter specification is: $a = 0.1$, $T = h_0 = \sigma = k = 1$.

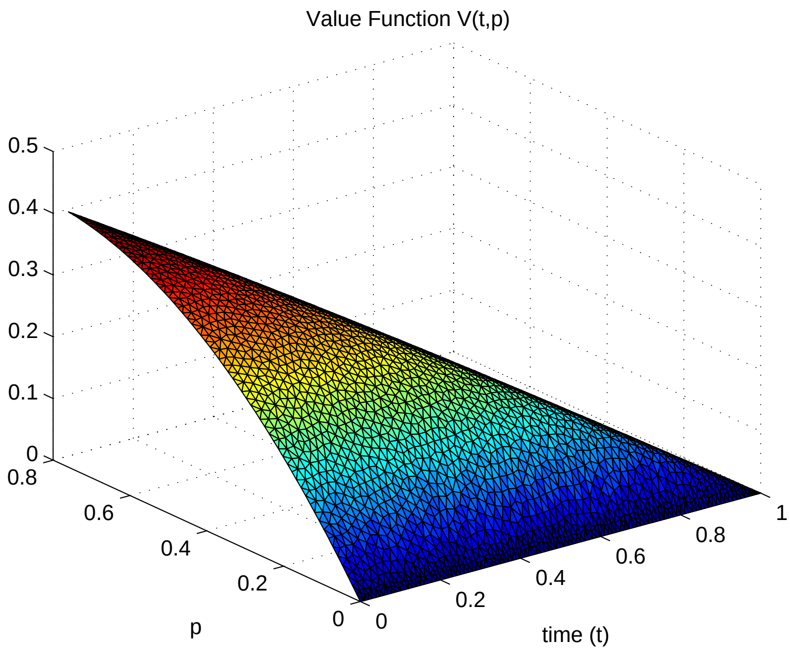

Figure 4: Optimal Policy: Pay-Performance Sensitivity $\beta(t,p)$

This figure plots pay-performance sensitivity $\beta(t,p)$ under the optimal contract based on the asymptotic analysis. The parameter specification is: $a = 0.1$, $T = h_0 = \sigma = k = 1$.

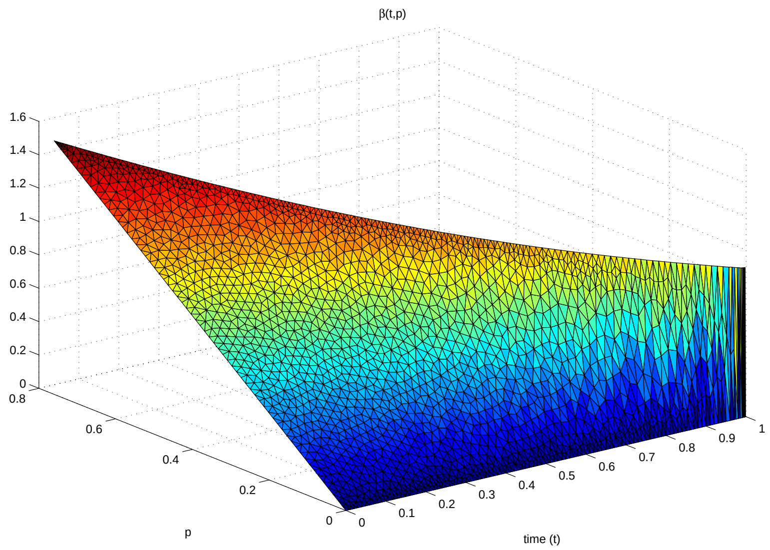

Figure 5: Optimal Policy: Randomization $\sigma^p(t,p)$

This figure plots the other optimal policy of the extent of randomization $\sigma^p(t,p)$ under the optimal contract based on the asymptotic analysis. The parameter specification is: $a = 0.1$, $T = h_0 = \sigma = k = 1$.

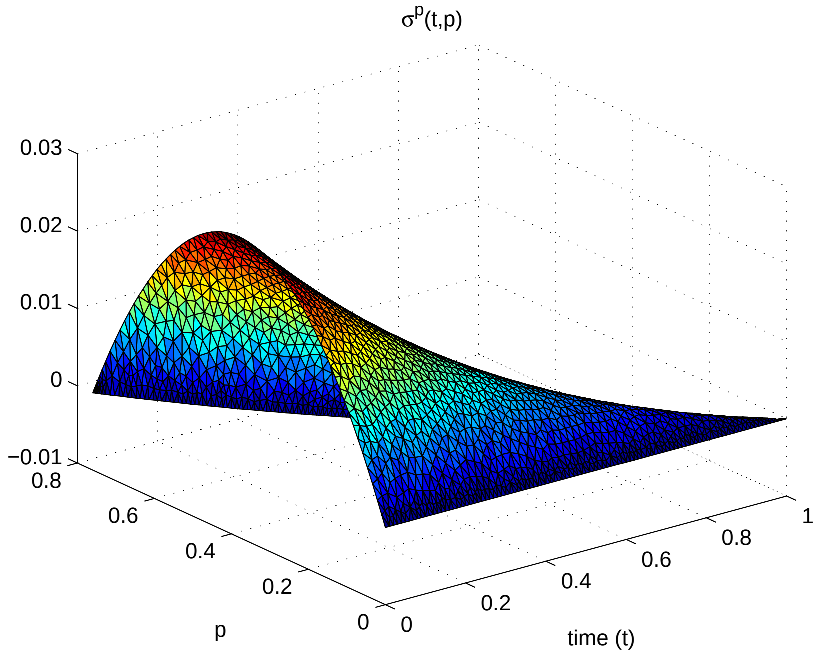

Figure 6: Comparative Statics with respect to Prior Variance (h_0)

Figure 6 (A) and (B) plots $\sigma^p(0,p)$ and $V(0,p)$ at time 0 for three possible values of h_0: 0.5, 1 (baseline), and 2, respectively. The other parameter values are given by: $a = 1$, $k = \sigma = 1$.

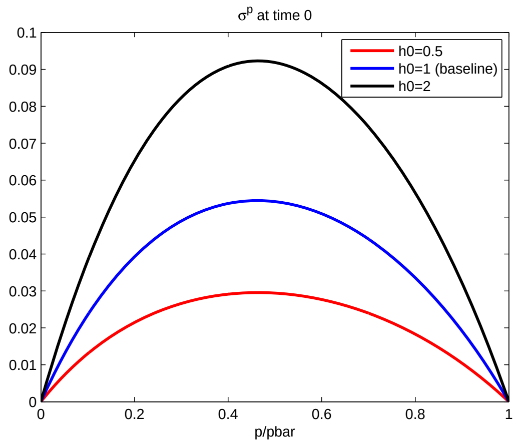

Figure 6 (A)

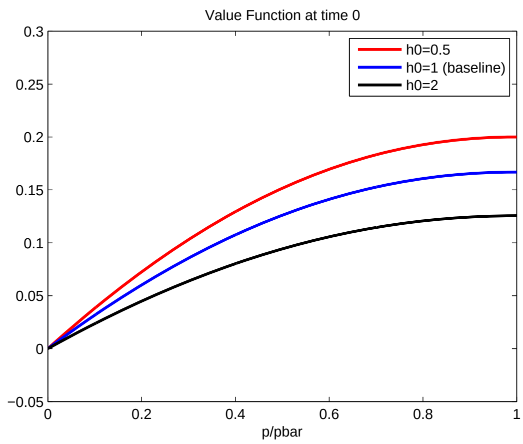

Figure 6 (B)

Figure 7: Comparative Statics with respect to Risk Aversion (a)

Figure 7 (A) and (B) plots $\sigma^p(0,p)$ and $V(0,p)$ at time 0 for three possible values of a: 0.1, 0.5, 1 (baseline), 1.5 and 2, respectively. The other parameter values are given by: $h_0 = 1$, $k = \sigma = 1$.

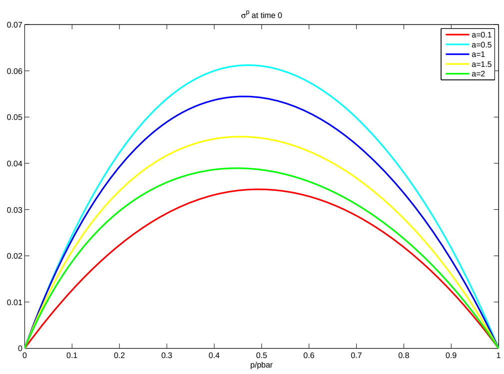

Figure 7 (A)

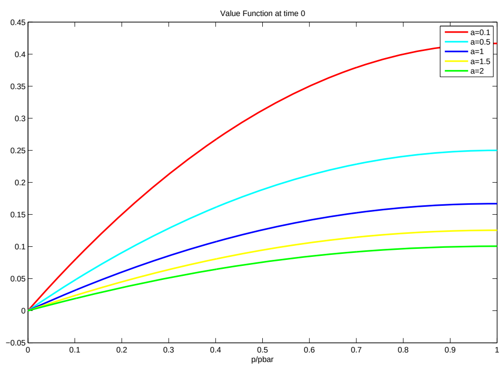

Figure 7 (B)

Figure 8: Simulation

Figure 8 plots one simulation path of the equilibrium outcomes. Subplot (1,1) plots the simulated effort path in the general case (blue solid lines) vs in the benchmark case (red dashed lines). Subplot (1,2) plots the simulated changes in incentives (multiplied by 10, blude solide lines) vs the changes in output (green dashed lines). Subplot(2,1) plots the simulated incentive path (i.e. β_t) in the general case (blue solid lines) vs in the benchmark case (red dashed lines). Subplot (2,2) plots the state variable (i.e., p_t or information rent) path in the general case (blue solid lines) vs in the benchmark case (red dashed lines). The parameter values are given by: $a = \sigma = h_0 = k = 1$ and $m_0 = 0$.