A Market Microsructure Theory of the Term Structure of Asset Returns

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Big Picture Question

Returns are unpredictable:

- Hayek (1945): The market is the machine that aggregates information correctly.
- Muth (1961): The rational expectations equilibrium means that prices are consistent with the relevant theory.
- Fama (1970): The market efficiency similarly suggests that it is difficult to make money.

There is a lot of empirical evidence, however, that returns are predictable, for example, see Jegadeesh and Titman (1993), DeBondt and Thaler (1985), Shiller (1981).

Main Idea

We derive **a structural model** of the expected returns based on the dynamic model of trading among oligopolistic traders who trade because of heterogeneous beliefs and heterogeneous information.

We show that the market does not aggregate information perfectly, but instead equilibrium returns are predictable and depend on the entire history of dividend-to-price ratios and dividends.

Term structure of returns is consistent with empirical evidence.

The Three Sets of Parameters

To derive empirical implications of the model and to understand how returns predictability may arise, we need to think about interaction of the three sets of parameters:

- Parameters that are correct according to **traders**;
- Parameters that are correct according to **the external observer** (economist/econometrician);
- Parameters that are correct according to **the representative agent** who aggregates behavior of traders.

Roadmap

- First, we use the dynamic smooth trading model of Kyle, Obizhaeva and Wang (2013) for modeling the behavior of individual investors and show that the incorrect beliefs of the representative agent may naturally arise in the equilibrium due to information aggregation.

- Second, we show how predictability of returns arises when prices are set by a representative agent (who aggregates behavior of traders) with possibly incorrect beliefs about the precision of privately observed signals.

The Three Motivating Examples

The examples illustrate several principles which are important in this paper.

They are all based on modeling market prices as the result of a single representative agent processing information.

Main message: The use of incorrect parameters by the market provides an explanation for some asset pricing puzzles such as risk premium puzzle of of Mehra and Prescott (1985) and excess volatility puzzle of Shiller (1981).

#1: Model With GBM Dividends

Dividends follow the process

$$dD(t) = \gamma \cdot D(t) \cdot dt + \sigma \cdot D(t) \cdot dB(t).$$

The dividend growth rate is γ. The price is $P(t) = D(t)/(r - \gamma)$.

Suppose the true dividend growth rate is $\hat{\gamma}$. From the economist's perspective the expected return is:

$$\hat{E}_t \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = (r - \gamma + \hat{\gamma}) \cdot P(t).$$

From the market's perspective the expected return is $r \cdot P(t)$. The volatility of returns σ is unaffected by incorrect beliefs.

#2: Model With Arithmetic AR-1 Dividends

Dividends follow the process

$$dD(t) = -\alpha \cdot (D(t) - \bar{D}) \cdot dt + \sigma \cdot dB(t).$$

The mean-reversion parameter is α. The price is $P(t) = \frac{\bar{D}}{r} + \frac{D(t) - \bar{D}}{r + \alpha}$.

Suppose the true mean-reversion parameter is $\hat{\alpha}$. From the economist's perspective, the expected return is:

$$\hat{E}_t \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = r \cdot P(t) + \frac{\alpha - \hat{\alpha}}{r + \alpha} \cdot (D(t) - \bar{D}).$$

From the market's perspective the expected return is $r \cdot P(t)$. The volatility of returns $\sigma \cdot (r + \alpha)^{-1}$ depends on beliefs of the market but not the true beliefs.

#3: One-Period Model of Info Processing

An asset has a liquidation value $v \sim N(0,1)$ at time 2. Investors observe a signal $\Delta I = \tau^{1/2} v + z$ with $z \sim N(0,1)$ and trade at time 1. The market thinks the precision is τ; its true value is $\hat{\tau}$.

For period from $t = 0$ to $t = 1$:

$$\hat{E}\{P_1 - P_0 | \Delta I\} = \frac{\tau^{1/2}}{1+\tau} \Delta I, \quad \hat{V}^{1/2}\{P_1 - P_0\} = \frac{\tau^{1/2}(1+\hat{\tau})^{1/2}}{1+\tau}.$$

For period from $t = 1$ to $t = 2$:

$$\hat{E}\{v - P_1 | \Delta I\} = \left(\frac{\hat{\tau}^{1/2}}{1+\hat{\tau}} - \frac{\tau^{1/2}}{1+\tau} \right) \Delta I.$$

$$\hat{V}^{1/2}\{v - P_1\} = \frac{((1+\tau-\hat{\tau}^{1/2}\tau^{1/2})^2 + \tau)^{1/2}}{1+\tau}.$$

Summary for Motivating Examples

#1: Pessimistic beliefs about the growth rate of dividends lead to a higher expected return (the equity premium puzzle). Risk premia can be counter-cyclical, if investors' beliefs are too pessimistic in recessions and too optimistic in booms;

#2: Incorrect belief that mean-reverting dividends are too persistent leads to excess volatility and mean reversion in asset prices;

#3: Overconfidence about signals can lead to excess volatility and mean reversion. Dynamic steady-state models are better for studying dynamic properties of returns.

Lessons Learned from Motivating Examples

- The actual returns process depends on two sets of parameters: correct parameters and possibly incorrect parameters used by the market.
- The use of incorrect parameters by the market can affect expected return, returns volatility, and the entire term structure of expected returns.
- It is usually more appropriate to model financial markets using dynamic steady-state models, because insights of static non-stationary models often can not be easily mapped into real data.

Shortcomings of Motivating Examples

The examples do not illustrate how overconfident processing of private information works in a dynamic context, nor do the examples show how information aggregation in prices results from the dynamic trading decisions of individual market participants.

Plan for the Rest of Paper

In the rest of this paper we extend these examples in two steps:

First, we model the behavior of individual investors and show that the incorrect beliefs of the representative agent may naturally arise in the equilibrium due to information aggregation.

Second, we show how predictability of returns arises when prices are set by a representative agent with possibly incorrect beliefs.

Smooth Trading Model of KOW (2013): Overview

KOW (2013) present a dynamic market microstructure model of speculative trading among oligopolistic traders, who agree to disagree about precision of private information.

- **Overconfidence:** Trade based on “relative” agreement to disagree. Each trader believes his flow of information is more precise than other traders believe it to be.
- **Market Power:** Each trader learns from prices and restricts trading to reduce impact.
- **Symmetry:** No noise traders; no market makers; no rational uninformed traders.

The model is a fully-fledged dynamic version of Kyle (1989), but trading is based on **agreement-to-disagree**, not noise trading.

KOW (2013): Assumptions

- There are N risk-averse oligopolistic traders, who trade a risky asset with a zero net supply against a risk-free asset.
- Risk-free rate is r.
- A risky asset pays out dividends $D(t)$ growing at a rate $G^*(t)$.

$$dD(t) = -\alpha_D \cdot D(t) \cdot dt + G^*(t) \cdot dt + \sigma_D \cdot dB_D,$$

$$dG^*(t) = -\alpha_G \cdot G^*(t) \cdot dt + \sigma_G \cdot dB_G.$$

Dividends D are observable. Growth rate G^* is unobservable.

KOW (2013): Information Flow

- Let $E^n_t\{\dots\}$ and $var^n_t\{\dots\}$ use all information and n's beliefs:

$$G_n(t) := E^n_t\{G^*(t)\}, \quad \Omega := var^n_t\left\{\frac{G^*(t)}{\sigma_G}\right\}$$

- Each trader observes the **public information** flow $dI_0(t)$ from dividends ($\tau_0 = \Omega \cdot \sigma_G^2/\sigma_D^2$):

$$dI_0(t) = \tau_0^{1/2} \cdot \frac{G^*(t)}{\Omega^{1/2}\sigma_G} \cdot dt + dB_0.$$

- Each trader n has continuous flow of **private information** $dI_n(t)$ about the unobserved growth rate $G^*(t)$:

$$dI_n(t) = \tau_n^{1/2} \cdot \frac{G^*(t)}{\Omega^{1/2}\sigma_G} \cdot dt + dB_n, \qquad n = 1, \dots, N,$$

- Trader n infers from prices the **average of other signals**

$$I_{-n} := \frac{1}{N-1}\Sigma_{m \neq n} I_m(t).$$

KOW (2013): Bayesian Updating

- Each trader n thinks that his own information has high precision $\tau_n = \tau_H$ and other traders have low precision $\tau_m = \tau_L$, $m \neq n$, with $\tau_H > \tau_L \geq 0$.
- Trader n constructs signals $H_i, i = 0, ..N$ as weighted average of information flow:

$$H_n(t) := \int_{u=-\infty}^{t} e^{-(\alpha_G+\tau)\cdot(t-u)} \cdot dI_n(u),$$

$$\textit{where} \quad \tau = \tau_0 + \tau_H + (N-1)\cdot\tau_L, \qquad \Omega^{-1} = 2\cdot\alpha_G + \tau.$$

$$G_n(t) = \sigma_G \cdot \Omega^{1/2} \cdot \left(\tau_0^{1/2} \cdot H_0(t) + \tau_H^{1/2} \cdot H_n(t) + \sum_{m \neq n} \tau_L^{1/2} \cdot H_m(t) \right)$$

- The importance of each bit of information about G decays exponentially at a rate $\alpha_G + \tau$, the same for every trader.

KOW (2013): Optimization

Each trader n chooses consumption path c_t and trading rate x_t to maximize CARA utility function $U(c_t) = -e^{-A \cdot c_t}$,

$$V(M, S, D, H_0, H_n, H_{-n}) = \max_{\{c_t, x_t\}} \quad E_t \left[\int_{u=t}^{\infty} -e^{-\rho u} \cdot U(c_u) \cdot du \right].$$

Inventory: $dS(t) = x_t \cdot dt$.

Cash: $dM(t) = (r \cdot M(t) + S(t) \cdot D(t) - c_t - P(x_t) \cdot x_t) \cdot dt$.

Dividends: $dD(t) = -\alpha_D \cdot D(t) \cdot dt + G^*(t) \cdot dt + \sigma_D \cdot dB_D$.

Information Flow Dynamics: dH_n and $dH_{-n} = \sum_{m \neq n} dH_m$.

KOW (2013): Linear Flow Equilibrium

- Numerical results consistent with "derived" existence condition

$$\frac{\tau_H^{1/2}}{\tau_L^{1/2}} > 2 + \frac{2}{N-2}.$$

- Market clearing quantities are derivatives of inventories, implying partial adjustment towards "target inventory", depending on constant C_L,

$$x_n(t) = dS_n(t)/dt = \gamma_S \cdot \Big(C_L \cdot (H_n(t) - \frac{1}{N-1} \sum_{m \neq n} H_m(t)) - S_n(t) \Big).$$

- Equilibrium price is dampened average, i.e., $0 < C_G < 1$, of buy and hold valuations based on Gordon's growth formula:

$$P^*(t) = \frac{D(t)}{r + \alpha_D} + \frac{C_G \cdot \frac{1}{N} \sum_{n=1}^{N} G_n(t)}{(r + \alpha_D)(r + \alpha_G)}.$$

Dampening Effect

Even if all traders agree on the fundamental value today, each trader believes that other traders will find out their information is incorrect in the future. Despite fundamentals, each trader "agrees" on a dampened equilibrium price. This effect does not exist in one-period model.

(1936): *"It is not sensible to pay 25 for an investment of which you believe the prospective yield to justify value of 30, if you also believe that the market will value it at 20 three months hence."*

In our dynamic model, traders also seem to be preoccupied with **"short-term price dynamics"** rather than **"hold-to-maturity"** values. The market internalizes overconfidence of traders and corrects equilibrium prices, so instead price deviations are dampened.

The Representative Agent: Theorem

The information aggregation is consistent with the information processing by the representative agent, whose beliefs are defined by the three parameters, growth-rate persistency $\breve{\alpha}_G > \alpha_G$, growth-rate volatility $\breve{\sigma}_G$, and each signal's precision $\breve{\tau}_I$,

$$\breve{\tau}_I = \tau_p \cdot \frac{C_G \cdot (r + \alpha_G + \tau)}{r + \alpha_G + C_G \cdot (\tau_0 + N \cdot \tau_p)},$$

$$\breve{\alpha}_G = \alpha_G + \frac{r + \alpha_G}{r + \alpha_G + C_G \cdot (\tau_0 + N \cdot \tau_p)} \cdot (\tau - C_G \cdot (\tau_0 + N \cdot \tau_p)),$$

$$\breve{\sigma}_G = \sigma_G \cdot \left[\frac{C_G \cdot (r + \alpha_G + \tau)}{r + \alpha_G + C_G \cdot (\tau_0 + N \cdot \tau_p)} \cdot \left(1 + \frac{\breve{\alpha}_G - \alpha_G}{2 \cdot \alpha_G + \tau}\right)\right]^{1/2},$$

where $\tau_p^{1/2} := (\tau_H^{1/2} + (N-1)\tau_L^{1/2})/N$. The representative agent agrees with traders on the other parameters.

The Representative Agent: Proof

The representative agent would need to have beliefs such that the equilibrium price

$$P(t) = \frac{D(t)}{r+\alpha_D} + C_G \cdot \frac{\sigma_G \cdot \Omega^{1/2}}{(r+\alpha_D)(r+\alpha_G)} \cdot \left(\tau_0^{1/2} \cdot H_0(t) + \tau_p^{1/2} \cdot \sum_{n=1,..N} H_n(t)\right)$$

coincides with his estimate of the fundamental value

$$\breve{F}(t) = \frac{D(t)}{r+\alpha_D} + \frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r+\alpha_D)(r+\breve{\alpha}_G)} \cdot \left(\breve{\tau}_0^{1/2} \cdot \breve{H}_0(t) + \breve{\tau}_I^{1/2} \cdot \sum_{n=1,..N} \breve{H}_n(t)\right).$$

Signals $\breve{H}_n(t)$ and $H_n(t)$ have to coincide so $\breve{\alpha}_G + \breve{\tau} = \alpha_G + \tau$.

The Representative Agent: Intuition

- If $\breve{\alpha}_G = \alpha_G$ and $\breve{\sigma}_G = \sigma_G$, there are no symmetric beliefs that can simultaneously match both current price level and its dynamics.
- The representative agent will need to disagree with the market even with respect to parameters the traders themselves may agree with each other about (such as σ_G, α_G and τ).

Information Aggregation

- Hayek (1945) says the key problem for designing a rational economic order is that constantly changing information, necessary for making decisions, is dispersed among individuals, rather than readily available to a central planner.
- Even if people act in their own interests, the price system is the mechanism that will communicate information to others and help to bring about the outcome, which "might have been arrived at by one single mind possessing all the information."
- Our paper: A central planner may need to be assigned beliefs that are inconsistent with beliefs of traders in the model.

The Economist

Denote the true parameters $\hat{\alpha}_G$, $\hat{\sigma}_G$, and $\hat{\tau}_I$. The economist uses those parameters—which may differ from the market's parameters—but processes information in a similar way:

$$\hat{H}_n(t) := \int_{u=-\infty}^{t} e^{-(\hat{\alpha}_G+\hat{\tau})\cdot(t-u)} \cdot dI_n(u), \quad n = 0, 1, \ldots N.$$

$$\hat{H}(t) = \hat{\tau}_0^{1/2} \cdot \hat{H}_0(t) + \sum_{n=1}^{N} \hat{\tau}_I^{1/2} \cdot \hat{H}_n(t).$$

$$\textit{where} \quad \hat{\tau} := \hat{\tau}_0 + \sum_{n=1}^{N} \hat{\tau}_I, \qquad \hat{\Omega}^{-1} = 2 \cdot \hat{\alpha}_G + \hat{\tau}.$$

$$\hat{G} := \hat{E}\{G^*(t)\} = \hat{\sigma}_G \cdot \hat{\Omega}^{1/2} \cdot \hat{H}(t).$$

The Market's and Economist's Signals

Both the market and the economist construct their signals $\breve{H}$ and $\hat{H}$ as the exponentially weighted average of past information. There is the following relationship:

$$\hat{H}_n(t) = \breve{H}_n(t) + (\breve{\alpha}_G + \breve{\tau} - \hat{\alpha}_G - \hat{\tau}) \cdot \int_{k=-\infty}^{t} e^{-(\hat{\alpha}_G + \hat{\tau}) \cdot (t-k)} \cdot \breve{H}_n(k) \cdot dk.$$

If the market has correct beliefs, then $\hat{H}_n(t) = \breve{H}_n(t)$. Otherwise, the relationship depends on the entire history of information flow. The economist extracts information from the market prices and re-weight it according to his scheme.

The Expected Returns

The returns depend both on the traders' parameters determining prices and the economist's parameters defining dynamics:

$$\hat{E}_t \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = r \cdot P(t) + a \cdot H(t) + b \cdot \hat{H}(t),$$

where a and b are coefficients.

- When the market uses correct parameters, then it obtains the expected return $r \cdot P(t)$.
- Otherwise, the market also obtains an unexpected risk premium $a \cdot H(t) + b \cdot \hat{H}(t)$.
- The volatility is constant depending on both sets of parameters.

Returns Related to Past Prices and Dividends

The equilibrium returns follow a complicated path-dependent and auto-correlated process:

$$
\begin{aligned}
dP(t) + D(t)\cdot dt &= r\cdot P(t)\cdot dt + \alpha_1\cdot\Big(P(t) - \frac{D(t)}{r+\alpha_D}\Big)\cdot dt \\
&+\alpha_2\cdot\Big[\int_{u=-\infty}^{t}\Big(P(u) - \frac{D(u)}{r+\alpha_D}\Big)\cdot e^{-(\hat{\alpha}_G+\hat{\tau})(t-u)}\cdot du\Big]\cdot dt \\
&+\alpha_3\cdot\Big[\int_{u=-\infty}^{t} e^{-(\hat{\alpha}_G+\hat{\tau})\cdot(t-u)}\cdot dI_0(u)\Big]\cdot dt + d\hat{B}_r^*(t),
\end{aligned}
$$

Traders obtain the expected return of $r\cdot P(t)$ and an unexpected return related to the history of deviations of prices from their long-term mean and dividends surprises.

The Term Structure of Returns

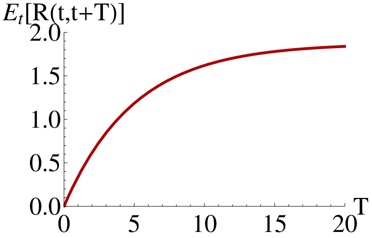
The term structure of returns $R(t, t+T)$ at time t can be represented as a linear combination of the average traders' sufficient statistics $H(t)$ and the economist's sufficient statistics $\hat{H}(t)$, inferred from past prices and dividends:

$$R(t, t+T) = \beta_1(T) \cdot H(t) + \beta_2(T) \cdot \hat{H}(t) + \bar{B}(t, t+T),$$

where $\beta_1(T)$ and $\beta_2(T)$ are coefficients. The term structure of returns exhibits different patterns depending on the values of parameters.

Numerical Example #1

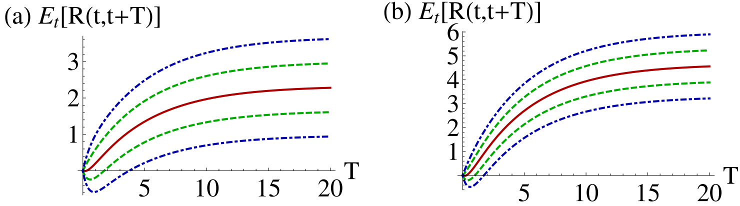
Parameters: $\hat{\alpha}_G = \alpha_G$, $\hat{\sigma}_G = \sigma_G$, and $\hat{\tau} = \tau$; state variables $\hat{H}(t) = H(t) = +1$ std.



There is momentum in returns due to the dampening effect of the Keynsian beauty contest. The cumulative returns level up as the profit opportunities disappear with time.

Numerical Example #2

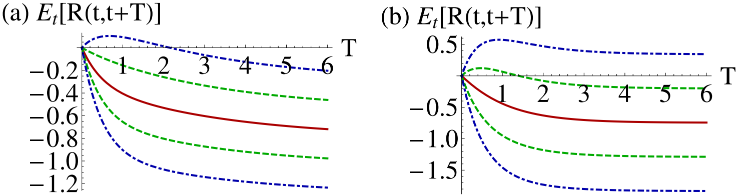
Parameters: $\hat{\alpha}_G = \alpha_G$, $\hat{\sigma}_G = \sigma_G$, but $\hat{\tau} < \tau$; state variables (a) $H(t) = +1$ std and (b) $H(t) = +2$ std.



The momentum effect continues to be dominating. But when signals of the economist are very low, then the returns exhibit slight mean-reversion in the short run.

Numerical Example #3

Parameters: $\hat{\alpha}_G > \alpha_G$, $\hat{\sigma}_G < \sigma_G$, but $\hat{\tau} < \tau$; state variables (a) $H(t) = +1$ std and (b) $H(t) = +2$ std.



The figure depicts more realistic patterns. The return exhibits momentum in the short run and mean-reversion in the long run.

Summary for Numerical Examples

The term structure of returns exhibits different patterns depending on the parameters, but it has particular features:

- The graph always starts from zero and converges to a constant as the horizon increases;
- The derivative of $R(t, t+T)$ with respect to T does not change its sign more than once;
- Thus, there are four possible patterns: only momentum, only mean-reversion, first mean-reversion and then momentum, first momentum and then mean-reversion.

Conclusion

- Information aggregation in markets with heterogeneous traders may result in incorrect beliefs of the representative agent and returns predictability.
- In our structural model, excess returns are time varying. They depend on the entire history of past prices and dividends, being broadly consistent with documented anomalies.
- The calibration of the model and studying whether it may generate quantitatively realistic predictions on horizons and magnitudes of anomalies are interesting future research questions.