Do Short Selling Constraints Have Price Effect?

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Motivating Question

Do short selling have price effect?
Stabilizing, destabilizing, or no impact?

The Literature

- Lintner (1970): no price effect; homo-belief
- Miller (1977): upward bias when short sales are prohibited; single asset; hetero-beliefs
- Jarrow (1983): multi-asset; Slutsky effect; hetero-beliefs
- Ma, Hu and Xu (2014): derivative trading; price uncertainty; homo- or hetero- beliefs; upward bias if agreeing on volatility . . .

Hidden Assumption: Short orders can be always executed

The Literature (Cont.)

- Duffie et al (2002): Price bubble when short-selling constraints are weakened
  - Single asset, which is hard to borrow
  - Non-competitive security lending market
  - Lenders have more bargaining power over the lending fee
- Kaplan et al. (2013): Experimental finding that shorting supply shock has NO impact on stock returns
- Wang, Ma and Sun (2014): single asset; market segmentation; competitive security lending market; upward bias or sticky price effect, depending on . . .

Outline

Model Setup

- No restriction on borrowing and lending of the risk free asset
- No taxes and transaction costs
- Investors face short selling constraints: margin deposit equals to $m_{kj}P_j$. The cash deposit earns no interest.
- Investor's belief: μ_k and $\Sigma_k = \left(\sigma_{ij}^k\right)_{J \times J}$.
- Investors' endowment in shares: $\mathcal{N}_j^k \geq 0, j = 0, 1, \cdots, J$.
- Investor's preference over end of period wealth W is summarized by a mean-variance utility function

$$u_k(W) = \mathbb{E}_k[W] - \frac{\alpha_k}{2}\mathrm{Var}_k[W],$$

where α_k measures investor k's degree of risk aversion.

Model Setup

- Let ϕ_0 be position in the risk free asset.
- Let $\phi \in \mathbb{R}^J$ be a risky portfolio. We may write $\phi = \phi_+ - \phi_-$, where $\phi_+ \geq 0$, $\phi_- \geq 0$ and $\phi_+ \cdot \phi_- = 0$.
- Denote by $m_k = (m_{kj})$ the margin rule facing by investor k.
- The (minimum) cash deposit for short one unit of security j is $m_{kj} P_j$, and that it earns no interest.
- The end of period payoff resulting from selling short one unit of security j:

$$(1 + m_{kj}) P_j - X_j = m_{kj} P_j + P_j - X_j$$

in which $m_{kj} P_j$ is the redemption of the cash deposit, and $P_j - X_j$ is the net proceed resulting from short sale.

Notations

- We write $\Sigma = \langle\sigma\rangle \Gamma \langle\sigma\rangle$, $\Gamma = \left[\rho_{ij}\right]$ is correlation matrix, σ is volatility vector, and $\langle\sigma\rangle$ is the diagonal matrix associated with σ.
- Introduce two Sharpe ratios

$$\pi \triangleq \langle\sigma\rangle^{-1} \left[\mu - (1+r) P\right]$$
$$\overline{\pi} \triangleq \langle\sigma\rangle^{-1} \left[\mu - \left(\overline{1} - rm\right) \boxdot P\right]$$

  Here, $x \boxdot y$ is an vector with j element $x_j y_j$.
- $\overline{\pi}_j - \pi_j = r \frac{(1+m_j)P_j}{\sigma_j} \geq 0$ measures additional "cost of capital" on short selling.

Optimal Choice Problem

With $\phi \longrightarrow \frac{\langle\sigma\rangle^{-1}\varphi}{\alpha}$ investor's optimal choice problem becomes

$$\max_{\varphi\in\mathbb{R}^J} -(\overline{\pi}-\pi)^\intercal\varphi_- + \pi^\intercal\varphi - \frac{1}{2}\varphi^\intercal\Gamma\varphi$$

- To investors who share the same beliefs and margins, their optimal risky portfolios must be proportional to each other.
- To investors who share a common correlation matrix and Sharpe ratios, they must end up with common LS division.

FOC

- In the absence of short selling restrictions ($m = -\overline{1}$), the optimal solution is given by the Markowitz portfolio $\varphi^{\mathrm{M}} \triangleq \Gamma^{-1}\pi$.
- In general, the first order condition becomes

$$\varphi \in \varphi^{\mathrm{M}} + \Gamma^{-1}\langle \overline{\pi} - \pi \rangle \left[1\left(\varphi_j\right) \right]_{J\times 1} \tag{1}$$

in which

$$1(x) \triangleq \begin{cases} 0, & x > 0 \\ [0,1], & x = 0 \\ 1, & x < 0 \end{cases}.$$

- Markowitz portfolio φ^{M} is optimal if it is non-negative.
- When risky payoffs are uncorrelated, the optimal solution is given by $\varphi^* = \pi_+ - \overline{\pi}_-$.

LS-Division and Optimal Portfolio

Theorem

$J = T \cup N$, $T = L \cup S$ constitutes an optimal L-S division if and only if conditions 1 and 2 are simultaneously satisfied:

1. $\langle \overline{\pi}_N - \pi_N \rangle^{-1} \left(\Gamma_{NT} \varphi_T^* - \pi_N \right)$ *is* $[0,1]^N$*-valued.*
2. $\varphi_T^* \triangleq \Gamma_{TT}^{-1} \begin{bmatrix} \pi_L \\ \overline{\pi}_S \end{bmatrix}$ *are strictly positive for* $j \in L$*, and strictly negative for* $j \in S$

The optimal solution is given by

$$\varphi_N^* = \emptyset, \varphi_T^* = \Gamma_{TT}^{-1} \begin{bmatrix} \pi_L \\ \overline{\pi}_S \end{bmatrix} \tag{2}$$

Equilibrium

The equilibrium is a price vector $P \in \mathbb{R}^J$ and portfolio holdings $\{\phi_k, k = 1, \cdots, K\}$ that satisfy conditions 1 and 2 below:

1. Given P, for all k, ϕ_k is optimal to investor k.
2. Market clears for each risky assets; that is, $\sum_{k=1}^{K} \phi_k = \mathcal{N} \triangleq \sum_{k=1}^{K} \mathcal{N}^k$.

Equilibrium

Condition M: For each j there exists some agents k such that $rm_{kj} < 1$.

Let

$$\begin{aligned}
\overline{\alpha}^{-1} &= \sum_{k \in K} \alpha_k^{-1} \\
\overline{\alpha}^{-1}\Sigma^{-1} &= \sum_{k \in K} \alpha_k^{-1}\Sigma_k^{-1} \\
\overline{\alpha}^{-1}\Sigma^{-1}\mu &= \sum_{k \in K} \alpha_k^{-1}\Sigma_k^{-1}\mu^k \\
\overline{\alpha}^{-1}\Sigma^{-1}\Lambda &= \sum_{k \in K} \alpha_k^{-1}\Sigma_k^{-1}\Lambda_k
\end{aligned}$$

in which $\Lambda_k =$diag$\left[\left(\overline{1} + m_k\right) \boxdot \lambda_k\right]$ for $\lambda_k \in [0,1]^J$.

Equilibrium Characterization Theorem

Define correspondence $\varphi : [0,1]^{J\times K} \mapsto \mathbb{R}^{J\times K}$ with

$$\varphi_k(\lambda) = \Sigma_k^{-1}\mu_k - \Sigma_k^{-1}\left(I - \frac{r}{1+r}\Lambda_k\right) \times \left(I - \frac{r}{1+r}\Lambda\right)^{-1}(\mu - \overline{\alpha}\Sigma\mathcal{N})$$

Theorem

Under condition M, the equilibrium exists if and only if the set of fixed points to

$$\lambda_k \in 1(\varphi_k(\lambda)) \;\; \textit{for all } k$$

is non-empty. Moreover, each fixed point λ corresponds to an equilibrium outcome that is given by

$$\begin{cases} P = [(1+r)I - r\Lambda]^{-1}(\mu - \overline{\alpha}\Sigma\mathcal{N}) \\ \phi_k = \alpha_k^{-1}\varphi_k(\lambda), k \in K. \end{cases}$$

Existence of Equilibrium

Theorem

Under condition M, the range of equilibrium security prices is non-empty and forms a closed set in the Euclidean space $\mathbb{R}^J$.

Case 1. Homogeneous Beliefs and Margins

- Homogeneous beliefs
- Investors face common margin rules: margin ratio m_j may vary across securities.
- Risky securities are divided into two broad categories: primitives (J_+) and derivatives (J_0).

Equilibrium Outcome

The Sharpe-Lintner CAPM price is denoted by

$$P^* \triangleq (1+r)^{-1}\left[\mu - \overline{\alpha}\Sigma\mathcal{N}\right] \qquad (3)$$

Theorem

Assume that $rm_j < 1$ for all $j \in J_0$ and that $J_0 \neq \varnothing$. The economy contains a continuum range of equilibrium price $[P^*, P^* + \epsilon]$ with CAPM as a lower bound, where $\epsilon_j = 0$ for $j \in J_+$ and $\epsilon_j = \frac{r(1+m_j)}{1-rm_j} P_j^*$ for $j \in J_0$.

Observations

1. For primitive securities J_+, the equilibrium prices are uniquely given by the CAPM. Particularly, the equilibrium price for the market portfolio is uniquely given by the CAPM as well; that is, $P_a = \sum_{j \in J_+} \mathcal{N}_j P_j^*$.
2. For derivative securities J_0, there is an range of equilibrium price $\left[P_j^*, \frac{1+r}{1-rm_j} P_j^*\right]$ with CAPM as a lower bound. The degree of price indeterminacy increases in m_j and r.
3. An separation between "risk" and "model uncertainty" effect of asset pricing is obtained. While the CAPM price (P^*) summarizes all risk effect, the margin ratio m, along with the risk free interest rate r, determines the degree of model uncertainty.

Observations

4. Price indeterminacy prevails for "structured products" that involve derivative trading. Precisely, for a structured product that admits a portfolio decomposition into $\phi = \phi_+ - \phi_-$, its price range $\left[\underline{P}_\phi, \overline{P}_\phi\right]$ becomes

$$\underline{P}_\phi = \sum_{j \in J} \left(m_j \phi_{j-} + \phi_{j+} \right) P_j^*$$

$$\overline{P}_\phi = \underline{P}_\phi + \sum_{j \in J_0} \frac{1 + m_j}{1 - r m_j} \left(m_j \phi_{j-} + \phi_{j+} \right) P_j^*$$

Case 2. Heterogeneous margins

Heterogeneity in margins may potentially cause investors hold different mutual funds. It is thus not obvious if the CAPM will constitute an equilibrium outcome. Yet, Theorem stated below confirms the robustness of CAPM as an equilibrium outcome.

Theorem

Consider an economy with mean-variance investors of homogeneous beliefs (μ, Σ) *with possibly different degree of risk aversion* (α_k) *and margin rules* (m_{kj}). *There exists an equilibrium at which (a) the CAPM holds; (b) investors would conduct no short selling, and (c) they would optimally hold a combination of the risk free bond and the market portfolio.*

Example

The equilibrium price uncertainty remains a robust phenomenon ...

- Two investors, two risky securities and a risk free asset.
- The risk aversion coefficients are $\alpha_1 = 1/2$ and $\alpha_2 = 2$.
- The risk free interest rate is $r > 0$.
- The margin ratios m_{kj} are such that $m_{kj} r < 1$.
- The payoffs for the two risky securities are such that

$$\begin{aligned} \mu &= \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \\ \rho &\in [-1, 1] \end{aligned}$$

- Investors' endowments in risky securities are $\mathcal{N}^1 = \mathcal{N}^2 = [0, 1]$.

Example (Cont)

Equilibrium outcome:

- The equilibrium price for security 2 is $P_2 = \frac{0.4}{1+r}$.
- There is a range of equilibrium price for security 1:

$$\left[P_1^*, \frac{1+r}{1-r\underline{m}_1}P_1^*\right]$$

in which $\underline{m}_1$ is the lower margin ratio between the two, and $P_1^* = \frac{1-0.8\rho}{1+r}$ corresponds to the CAPM price.

- Price indeterminacy disappears when either investors faces no restriction on short selling security 1.

Case 3. Common Volatility

- Suppose all investors share a common variance and covariance matrix, but may form different expectations on security returns.
- Let K_j^+, K_j^- and K_j^0 be the set of investors respectively taking long, short and null positions in security j.
- Let

$$\rho_j^- \triangleq \frac{\sum_{k \in K_j^-} \alpha_k^{-1}}{\sum_{k \in K} \alpha_k^{-1}}, m_j^- \triangleq \frac{\sum_{k \in K_j^-} \alpha_k^{-1} m_{kj}}{\sum_{k \in K_j^-} \alpha_k^{-1}};$$

$$\rho_j^0 \triangleq \frac{\sum_{k \in K_j^0} \alpha_k^{-1}}{\sum_{k \in K} \alpha_k^{-1}}, m_j^0 \triangleq \frac{\sum_{k \in K_j^0} \alpha_k^{-1} m_{kj}}{\sum_{k \in K_j^0} \alpha_k^{-1}}.$$

  - $1 - \rho_j^0$ is a liquidity measure.
  - $\rho_j^- \left(1 + m_j^-\right)$ and $\rho_j^0 \left(1 + m_j^0\right)$ are negative sentiment measures.

Equilibrium

We obtain the following assessment of equilibrium price $\left[\underline{P}_j, \overline{P}_j\right]$ with

$$\underline{P}_j = \frac{1}{1 - \frac{r}{1+r}\rho_j^- \left(1 + m_j^-\right)} P_j^*$$
$$\overline{P}_j = \frac{1}{1 - \frac{r}{1+r}\rho_j^- \left(1 + m_j^-\right) - \frac{r}{1+r}\rho_j^0 \left(1 + m_j^0\right)} P_j^*$$

Observations

1. If investors' sentiment is overwhelmingly positive at security j (i.e., $\rho_j^0 = \rho_j^- = 0$), its equilibrium price must be given by the CAPM price P_j^*.
2. If $\rho_j^- > 0$ and $m_j^- > -1$ and $\rho_j^- \left(1 + m_j^-\right) < 1 + \frac{1}{r}$, there is a upward bias from the CAPM, and, in this case, the minimum (relative) gap from the CAPM is

$$\frac{\underline{P}_j - P_j^*}{P_j^*} = \frac{r\rho_j^- \left(1 + m_j^-\right)}{1 + r - r\rho_j^- \left(1 + m_j^-\right)} > 0$$

   which increases in r, ρ_j^- and m_j^-.
3. There is no price uncertainty if $\rho_j^0 \left(1 + m_j^0\right) = 0$; in this case, the equilibrium price is $\underline{P}_j \geq P_j^*$.
4. There is price uncertainty if $\rho_j^0 \left(1 + m_j^0\right) > 0$; and, in this case, the degree of price uncertainty increases in $r, \rho_j^0, \rho_j^-, m_j^0$ and m_j^-.

Market Segmentations

Two types of investors: M potential lenders and N liquidity traders

- Potential lenders can long or short (if permitted), and can lend their shares to short sellers when take long positions.
- Liquidity traders can long or short (if permitted), but not permitted to lend their shares out.
- Investors may face short selling prohibition. Let a $(0 \leq a \leq 1)$ be the proportion banned from short selling. Thus, the total population mass of potential short sellers is

$$K = (1 - a)(M + N).$$

A Basic Model

- One risky security: The total shares outstanding is S
- Investors agree on the volatility, but differ in the expected payoffs. Their expectations are uniformly distributed on $[F - cF, F + cF]$.
  - F measures investors' aggregated sentiment, and
  - $c \in (0, 1)$ is the dispersion of opinion among the investors, thus measures the degree of heterogeneity in beliefs.
- The optimists potential lenders take long position at the spot market price P and may wish to lend out their shares to earn extra lending fee L.
- The pessimists (among all investors) who want to short become borrowers. To do so, the short sellers must pledge cash collateral exceeding the proceed of the sale by a margin ratio m. The minimum cash deposit per unit short sale is thus $P(1 + m)$.

A Basic Model (Cont.)

At the end of the trading session, the security payoff is realized.

- The short sellers must buy back their borrowed shares to cover their short position.
- The lenders must return the cash collateral to the short sellers, and rebate part of the interest (generated from the margin deposit) at a rebate rate $b \in (-\infty, r]$, where r is the fixed risk-free interest rate. This yields the lending fee:

$$L = P(1 + m)(r - b)$$

Twin Equilibrium

- Potential lender's optimal demand:

$$q_i = \lambda[V_i - P(1+r) + L]$$

where $\lambda \triangleq \frac{1}{\alpha\sigma^2}$ is investor's propensity to risk.

- Liquidity trader's optimal holding:

$$q_j = \begin{cases} \lambda[V_j - P(1+r)], & \text{if } V_j > P(1+r); \\ \lambda[V_j - P(1+r) + L)], & \text{if } V_j < P\,(1+r) - L; \\ 0, & \text{otherwise.} \end{cases}$$

Twin Equilibrium

Equilibrium is a pair $\{P^*, L^*\}$ such that both spot market and security lending market clear simultaneously.

- In equilibrium, optimistic potential lenders long position offsets short sellers' short position so long as the equilibrium lending fee is positive. In this case, the equilibrium security price is fully determined by the aggregated holdings of those optimistic liquidity traders.
- The equilibrium lending fee becomes zero when the security lending market (among security lenders) is highly competitive. In this case, the equilibrium security price will be determined by the market clearing condition in the security trading market.

Equilibrium Outcome

- The equilibrium lending fee is strictly positive if and only if

$$\left(1+\sqrt{\frac{M}{K}}\right)^2 \frac{S}{N} < \lambda c F \qquad \text{(A)}$$

- Condition A is satisfied when the population mass of potential lenders relative to that of potential short sellers is sufficiently low, and if the mass of liquidity traders is large, taking other environmental parameters as given.
- Taking market segmentations as given, condition A tends to be satisfied if investors' beliefs are highly dispersed, and if volatility is low, and investors are not too risk averse (i.e., $\lambda = \frac{1}{\alpha\sigma^2}$ is high).

Scenario A

Under Condition A, the equilibrium outcome is

$$P^* = \frac{1}{1+r}\left(F + cF - 2\sqrt{\frac{cF}{\lambda}\frac{S}{N}}\right)$$

$$L^* = 2\left(\frac{cF}{1+\sqrt{\frac{M}{K}}} - \sqrt{\frac{cF}{\lambda}\frac{S}{N}}\right)$$

$$SI^* = \frac{\lambda cF}{\left(\sqrt{\frac{S}{M}} + \sqrt{\frac{S}{K}}\right)^2}$$

Scenario B

When Condition A is violated, the equilibrium lending fee is zero ($L^* = 0$), and the equilibrium security price and short interest become

$$P^* = \frac{1}{1+r}\left(F - cF + 2\frac{cF - \frac{1}{\lambda}\frac{S}{M+N}}{1+\sqrt{1-a+\frac{a}{\lambda cF}\frac{S}{M+N}}}\right)$$

$$SI^* = \left(\frac{\sqrt{\frac{1}{\lambda cF}\frac{S}{M+N}} - 1/\sqrt{\frac{1}{\lambda cF}\frac{S}{M+N}}}{\frac{1}{\sqrt{1-a}} + \sqrt{1+\frac{a}{1-a}\frac{1}{\lambda cF}\frac{S}{M+N}}}\right)^2$$

The CAPM is obtained at $a = 0$ with

$$P^{\text{capm}} = \frac{1}{1+r}\left(F - \frac{1}{\lambda}\frac{S}{M+N}\right)$$

Observations

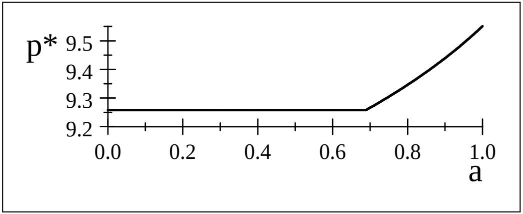
1. If there exist no market segmentations (i.e., $a = 0$, $N = \varnothing$), the CAPM constitutes the unique equilibrium. Otherwise, in either scenarios, the equilibrium price must be greater than the CAPM.
2. When short selling are banned to all investors $(a = 1)$, it yields Scenario B with

$$P^{\text{ban}} = \frac{1}{1+r}\left(F + cF - 2\sqrt{\frac{cF}{\lambda}\frac{S}{M+N}}\right)$$

This is strictly greater than the Scenario A price, and is above the CAPM.

Observations

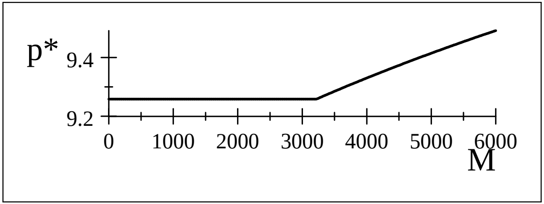
3. The effect of short selling prohibition (a) on the equilibrium outcome:
   - When a becomes sufficiently large $\Rightarrow$ condition A tends to be violated.
   - Scenario A: As a increases, the security price remains unchanged; the lending fee and the short interest decrease.
   - Scenario B: As a increases, the security price increases; the lending fee remains zero; the short interest decreases.



Price and Scope of Short Selling Prohibition

Observations

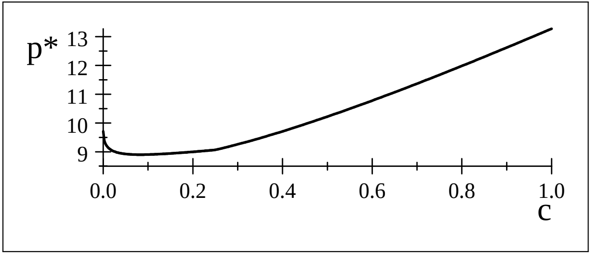
4. The population mass of potential lenders, M, measures the lending-side market segmentation.
   - A huge number of potential lenders can mitigate the market segmentation from Scenario A to Scenario B
   - Scenario A: As M increases, the security price remains unchanged; the lending fee decreases; the short interest increases.
   - Scenario B: As M increases, the security price increases; the lending fee remains unchanged; the short interest increases.



Price and Mass of Potential Lenders

Observations

5. The dispersion in beliefs c affects both sides of the markets positively.
   - When c is sufficiently large, an increase in c re-enforces the validity of condition A; otherwise, when c becomes sufficiently small, it mitigates to Scenario B.
   - Scenario A: As c increases, the security price, lending fee and short interest increase.
   - Scenario B: As c increases, the security price may decrease or increase; the lending fee remains zero; the short interest increases.



Price and Dispersion in Beliefs

Conclusions

Do short selling restrictions have price effect? and How?

We tend to answer "Yes" to the first part of the question.
No trivial answers to the second part.
Our theoretical research suggests that

- Short volumes, fees for short sales and security prices may not have logical causality relationships among each other. They are all endogenously affected by various environmental factors and policy variables such as the degree of market segmentations, the scope of short selling prohibition, the degree of heterogeneity in beliefs, ...
- Short selling and derivative trading, as two major aspects of financial innovations, together may have some strong implications on the well-being of the security market, and on the welfare of the society as a whole.

Thank You!