“Asset Pricing in a Large Economy”

Jeongmin Lee, Tao Li, and Mark Loewenstein

The Topic

- This talk is based on work with Jeongmin Lee (Wash U) and Tao Li (City U of HK).

- We develop a simple way to value cash flows when fundamentals are driven by a single state variable.

- We use this to examine how varying assumptions about fundamentals affect equilibrium.

- In particular, we study both the pricing of safe and risky assets.

- Very Very Preliminary

Some Related Literature

- Chan and Kogan (2002 JPE)
- Xiouros and Zapatero (2010 RFS)
- Parlour, Stanton, and Walden (2011, RFS)
- Loewenstein and Willard (2000, ET)(2000 JET) (2013 JET)
- Heston, Loewenstein, and Willard (2007 RFS)

The basic assumptions

- Uncertainty is generated by a d dimensional Brownian motion B_t defined on a probability space. $\mathcal{F}_t$ is the completed filtration generated by the Brownian motion. All processes below are assumed to satisfy appropriate measurability conditions.

- There is a continuum of agents each of whom has preferences given by

$$E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma_i}}{1-\gamma_i}dt\right] \tag{1}$$

- Markets are "complete."

- The aggregate endowment is given by $\delta(Y_t)$.

- $dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t$

- Y lives on the state space (l, r).

- Assume $E\left[\int_0^\infty e^{-\rho t}\delta(Y_t)dt\right] < \infty$.

Valuation

We want to evaluate

$$P_t = E\left[\int_t^{\infty} \frac{\pi_s}{\pi_t} c_t dt\right] \tag{2}$$

We will assume that $\pi_t = e^{-\rho t} g(Y_t)$ and $c_t = c(Y_t)$.

Valuation

Suppose we want to calculate

$$h(Y) = g(Y)P(Y) = E\left[\int_0^\infty e^{-\rho t} g(Y_t)c(Y_t)dt | Y_0 = Y\right] \quad (3)$$

Then $e^{-\rho t}h(Y_t) + \int_0^t e^{-\rho s} g(Y_s)c(Y_s)ds$ is a martingale.

Using Ito's Lemma we have the ODE

$$\frac{\sigma(Y)^2}{2} h''(Y) + \mu(Y)h'(Y) - \rho h(Y) = -g(Y)c(Y) \quad (4)$$

Solutions to the Homogeneous Equation

- There exists a non-negative decreasing solution $\Psi_1(Y)$ and a nonegative increasing solution $\Psi_2(Y)$.

- If l is unattainable $\lim_{Y \to l} \Psi_1(Y) = \infty$. If r is unattainable, $\lim_{Y \to r} \Psi_2(Y) = \infty$.

- In particular, if both boundaries are unattainable, then any non-negative solution to the homogeneous equation defines an asset pricing bubble.

The solution

Suppose the boundaries l and r are unattainable or reflecting boundaries. Then we have

$$P\left(Y_t\right)=\frac{\Psi_1\left(Y_t\right)}{g\left(Y_t\right)}\int_l^{Y_t}\frac{2g\left(\xi\right)c\left(\xi\right)\Psi_2\left(\xi\right)}{\sigma\left(\xi\right)^2\mathcal{W}\left(\Psi_1,\Psi_2\right)\left(\xi\right)}d\xi+\frac{\Psi_2\left(Y_t\right)}{g\left(Y_t\right)}\int_{Y_t}^{r}\frac{2g\left(\xi\right)c\left(\xi\right)\Psi_1\left(\xi\right)}{\sigma\left(\xi\right)^2\mathcal{W}\left(\Psi_1,\Psi_2\right)\left(\xi\right)}d\xi, \tag{5}$$

while if l is an attainable absorbing boundary and r is an unattainable boundary

$$P_t\equiv P\left(Y_t\right)=\frac{\Psi_1(Y)g(0)c(0)}{\Psi_1(l)g(Y)\rho}+$$

$$\frac{\Psi_1\left(Y_t\right)}{g\left(Y_t\right)}\int_l^{Y_t}\frac{2g\left(\xi\right)c\left(\xi\right)\Psi_2\left(\xi\right)}{\sigma\left(\xi\right)^2\mathcal{W}\left(\Psi_1,\Psi_2\right)\left(\xi\right)}d\xi+\frac{\Psi_2\left(Y_t\right)}{g\left(Y_t\right)}\int_{Y_t}^{r}\frac{2g\left(\xi\right)c\left(\xi\right)\Psi_1\left(\xi\right)}{\sigma\left(\xi\right)^2\mathcal{W}\left(\Psi_1,\Psi_2\right)\left(\xi\right)}d\xi, \tag{6}$$

Volatility

All bubble free streams of cash flow values have volatility bounded by the volatility of the homogeneous solutions.

$$\sigma_P(Y_t) = -\frac{g'(Y_t)}{g(Y_t)}\sigma(Y_t) + w(Y_t)\frac{\Psi_1'(Y_t)}{\Psi_1(Y_t)}\sigma(Y_t) + (1 - w(Y_t))\frac{\Psi_2'(Y_t)}{\Psi_2(Y_t)}\sigma(Y_t)$$

$$= -\frac{g'(Y_t)}{g(Y_t)}\sigma(Y_t) + \frac{\Psi_2'(Y_t)}{\Psi_2(Y_t)}\sigma(Y_t) + w(Y_t)\left(\frac{\Psi_1'(Y_t)}{\Psi_1(Y_t)} - \frac{\Psi_2'(Y_t)}{\Psi_2(Y_t)}\right)\sigma(Y_t), \quad (7)$$

$w(Y)$ is the fraction of the value from cash flows below Y.

Geometric Brownian Motion

Consider the state variable Y that is a geometric Brownian motion and satisfies the following stochastic differential equation

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t. \tag{8}$$

Then the homogeneous solutions for the fundamental ODE are Y^{β_1} and Y^{β_2}, where $\beta_1 > 0$ and $\beta_2 < 0$ solve the characteristic equation

$$\frac{\sigma^2}{2}\beta^2 + \left(\mu - \frac{\sigma^2}{2}\right)\beta - \rho = 0. \tag{9}$$

The equilibrium value of the cash flow $c(\cdot)$ is given by

$$P(Y) = \frac{2}{(\beta_1 - \beta_2)\,\sigma^2}\left[\frac{\int_Y^\infty \xi^{-\beta_1-1} g(\xi) c(\xi)\,d\xi}{Y^{-\beta_1} g(Y)} + \frac{\int_0^Y \xi^{-\beta_2-1} g(\xi)\,c(\xi)\,d\xi}{Y^{-\beta_2} g(Y)}\right]. \tag{10}$$

The volatility of the value of the cash flows is given by

$$\sigma_P(Y) = -\frac{g'(Y)}{g(Y)}\sigma Y + \beta_2\sigma + w(Y)(\beta_1 - \beta_2)\sigma, \tag{11}$$

The agents problem

We will consider solutions to the static problem of maximizing utility subject to the static budget constraint

$$E\left[\int_0^\infty \pi_t c_t dt\right] \leq W_0 \tag{12}$$

The solution, should it exist will be characterized by the first order conditions

$$e^{-\rho t} c_t^{-\gamma_i} = \frac{1}{\lambda_i}\pi_t \tag{13}$$

Equlibrium Allocation

- From the first order conditions we have

$$c_t^i = \left(\frac{\lambda_j}{\lambda_i} \right)^{-\frac{1}{\gamma_i}} \left(c_t^j \right)^{\frac{\gamma_j}{\gamma_i}} \tag{14}$$

- Assume that $\lambda_i = e^{a\gamma_i}$

- We assume the relative risk aversion coefficient γ_i is distributed according to the inverse-gamma distribution with shape parameter ϕ and scale parameter β, i.e., γ_i is distributed with the density

$$f(\gamma_i) = \beta^{\phi} \gamma_i^{-\phi-1} \exp\left(-\frac{\beta}{\gamma_i} \right) / \Gamma(\phi). \tag{15}$$

- We further assume $\forall j$, the weight assigned to each agent is exponential.

$$\lambda_j = e^{a\gamma_j}, \tag{16}$$

for some $a \in \Re$. This gives some flexibility in the resulting wealth distribution since we can weigh very risk averse agents less heavily than less risk averse agents by choosing $a < 0$ and vice-versa when $a > 0$.

- The resource constraint implies

$$\int_0^\infty c_j f\left(\gamma_j\right) dj = \int_0^\infty e^a \left(e^{-a} c_i\right)^{\gamma_i x} \frac{\beta^\phi e^{-\beta x} x^{\phi-1}}{\Gamma\left(\phi\right)} dx = \delta. \tag{17}$$

- Therefore, we have the following sharing rule

$$c_i = e^a \exp\left\{\frac{\beta}{\gamma_i}\left(1 - \frac{\delta\left(Y\right)^{-\frac{1}{\phi}}}{e^{-\frac{a}{\phi}}}\right)\right\}, \ \forall i. \tag{18}$$

- State price density $\pi_t = e^{-\rho t} g(Y_t)$ where

$$g\left(Y_t\right) = \left(\frac{c_{it}}{c_{i0}}\right)^{-\gamma_i} = \exp\left\{\beta\left(\frac{\delta\left(Y_t\right)^{-\frac{1}{\phi}} - \delta\left(Y_0\right)^{-\frac{1}{\phi}}}{e^{-\frac{a}{\phi}}}\right)\right\}. \tag{19}$$

Price of Risk

The price of risk (θ_t) is given by

$$\theta(Y_t) = -\frac{g'(Y_t)}{g(Y_t)}\sigma(Y_t) = \gamma_R(Y_t) \cdot \underbrace{\frac{\delta'(Y_t)}{\delta(Y_t)}\sigma(Y_t)}_{\text{vol. of agg. consumption}}, \tag{20}$$

where

$$\gamma_R(Y_t) = \frac{\beta e^{\frac{a}{\phi}}}{\phi}\delta(Y_t)^{-\frac{1}{\phi}}. \tag{21}$$

is a consumption weighted harmonic mean of risk aversion of agents.

Risk Free Rate

The locally riskless rate for borrowing and lending is given by

$$r(Y_t) = \rho + \gamma_R(Y_t)\frac{\delta'(Y_t)}{\delta(Y_t)}\mu(Y_t) - \frac{1}{2}\gamma_R(Y_t)\left(\gamma_R(Y_t) + 1 + \frac{1}{\phi}\right)\left(\frac{\delta'(Y_t)}{\delta(Y_t)}\right)^2\sigma(Y_t)^2 + \frac{1}{2}\gamma_R(Y_t)\frac{\delta''(Y_t)}{\delta(Y_t)}\sigma(Y_t)^2. \quad (22)$$

Caution

Proposition. *Suppose at each time t the aggregate endowment is lognormal distributed. Then every agent with $\gamma > 1$ will have negative infinite utility at the equilibrium allocation.*

This result says we cannot use $\delta(Y) = Y$ where Y is a geometric brownian motion or $\delta(Y) = \exp(Y)$ where Y is an Ornstein Uhlenbeck process.

A curious result

Proposition. *Suppose the aggregate endowment* $\delta(Y) = Y$ *where* Y *is given by a CEV process*

$$dY_t = \mu Y_t dt + \sigma Y_t^{\alpha} dB_t \tag{23}$$

where $\mu > 0$ *and* $\alpha > 1 + \frac{1}{2\phi}$. *Then the equilibrium with allocations given earlier exists. Moreover, these equilibria all have an equivalent martingale measure.*

This is suprising. The CEV process with $\alpha > 1$ has the property $E[Y_t] < e^{\mu t}$. $E[Y_t^p] < \infty$ for $p < 2\alpha - 1$. But $E[Y_t^p] = \infty$ for $p = 2\alpha$.

Curious Results

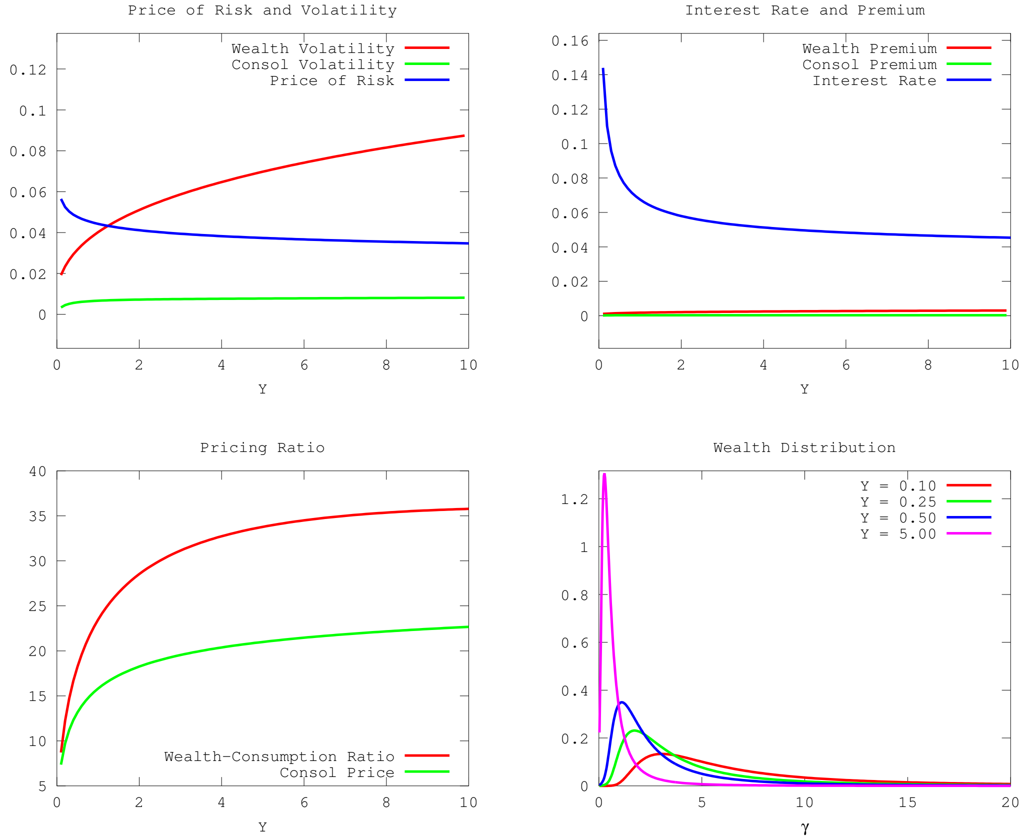
When $\alpha = 1 + \frac{1}{2\phi}$ the conditions for existence are more involved. However, in the special case where $\mu = \frac{\beta e^{\frac{a}{\phi}}}{2\phi}\sigma^2$ the equilibrium interest rate is constant.

In this case the value of the aggregate endowment is bounded by $\frac{Y}{\rho - \frac{\mu}{\phi}}$. But this has a bubble. Our solution to the ODE is lower.

The consol price is $\frac{1}{\rho - (1+\frac{1}{\phi})\mu}$.

Example

Figure 1: CEV



Model: $dY = \mu Y_t dt + \sigma Y_t^{\alpha} dB$, and $\delta(Y) = Y$. The model parameters are set as follows: $\mu = 0.02$, $\sigma = 0.03$, $\alpha = 1.45$, $\rho = 0.04$, $\phi = 1.8$, and $\beta = 10$.

Linear Drift Constant Volatility

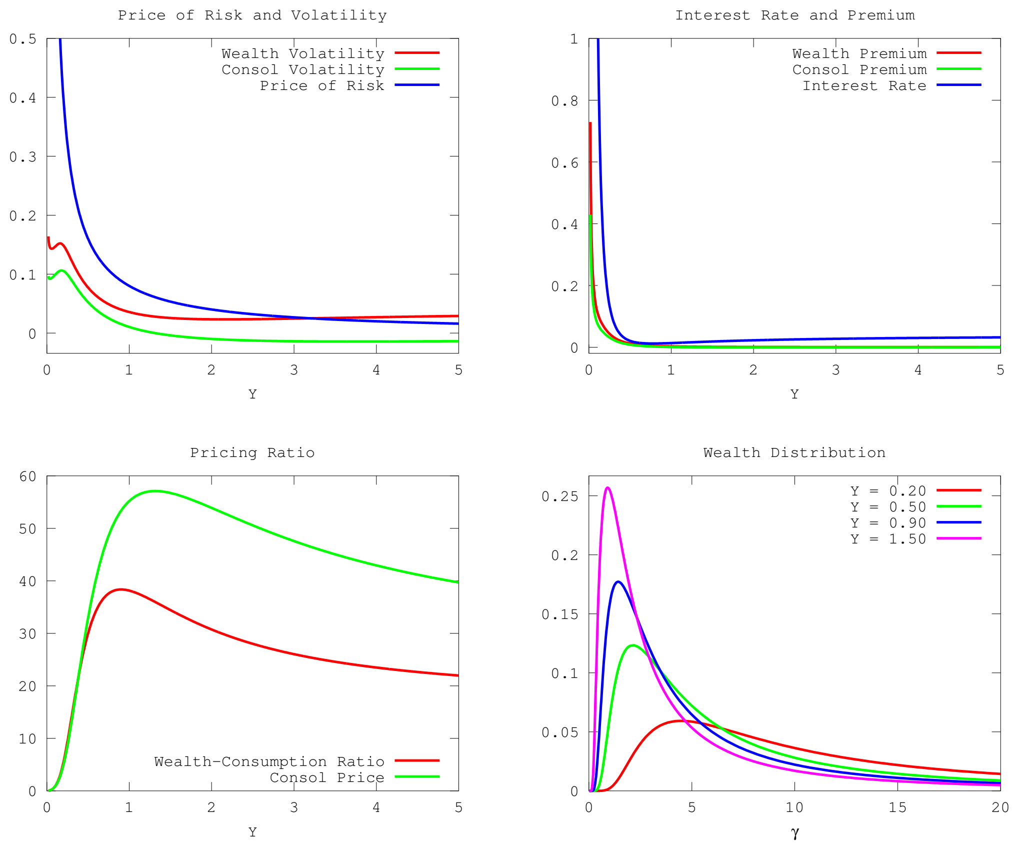
Suppose the aggregate endowment is given by

$$dY_t = (\kappa + \mu Y_t)dt + \sigma Y_t dB_t \tag{24}$$

and $\rho > \mu$. Then an equilibrium exists if $\phi > 1$. If $\phi = 1$ the existence conditions are more involved.

When $\phi = 1$ there may not exist an equaivalent martingale measure.

Figure 2: Linear Drift Constant Volatility



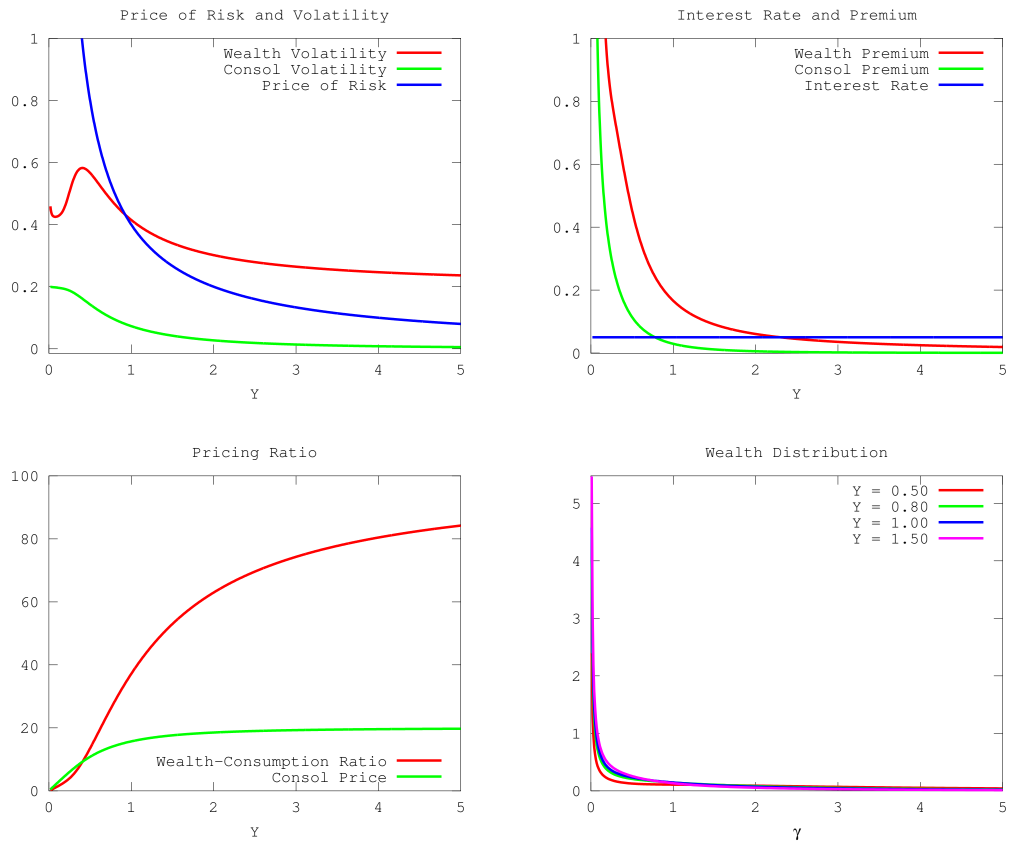
Model: $dY = (\kappa + \mu Y_t)dt + \sigma Y_t dB_t$, and $\delta(Y) = Y$. The model parameters are set as follows: $\rho = 0.04$, $\mu = -0.02$, $\sigma = 0.04$; $\phi = 1$, $\beta = 2$, $a = 0$, and $\kappa = 0.01$.

No Risk Neutral Measure

When $\phi = 1$, $\mu = \sigma^2$, and $\kappa = \beta e^{a} \frac{\sigma^2}{2}$, then the risk free rate is constant and equal to ρ.

But the value of the consol cash flows is lower than $\frac{1}{\rho}$ becuase there is no equivalent risk neutral measure.

Figure 3: Linear Drift Constant Volatility



Model: $dY = (\kappa + \mu Y_t)dt + \sigma Y_t dB_t$, and $\delta(Y) = Y$. The model parameters are set as follows: $\rho = 0.05$, $\mu = 0.04$, $\sigma = 0.2$; $\phi = 1$, $\beta = 2$, $a = 0$, and $\kappa = 0.04$.

What is wrong?

State price density $e^{-\rho t} g(Y_t) = e^{-\int_0^t r(Y_s)ds} \exp\left(-\frac{1}{2}\int_0^t \theta(Y_s)^2 ds - \int_0^t \theta(Y_s) dB_s\right)$.

$\exp\left(-\frac{1}{2}\int_0^t \theta(Y_s)^2 ds - \int_0^t \theta(Y_s) dB_s\right)$ is a local martingale but not a martingale.

If it were a martingale, it would define a change of measure where the state variable has dynamics

$$dY_t = (\kappa - \beta e^a \sigma^2 + \mu Y_t)dt + \sigma Y_t dB_t^Q \quad (25)$$

But this process can reach the origin while in the P measure it cannot. The two measures cannot be equivalent.

Pricing

We are only interested in pricing cash flows that have positive P probability. The Q measure will price things that have 0 P probability.

Think of the consol price under Q. It can be decomposed into the price of a consol which pays 1 until the state variable hits the origin and the price of a consol which pays 1 forever after the first time the state variable hits the origin.

Our formula gives the first component.

$$\frac{1}{\rho} - \frac{1}{\rho}\frac{\Gamma(\beta_2+1)}{\Gamma(\beta_2-\beta_1+1)} M\left(-\beta_1, \beta_2-\beta_1+1; \frac{2\hat{\kappa}}{\sigma^2 Y}\right)\left(\frac{-2\hat{\kappa}}{\sigma^2 Y}\right)^{-\beta_1} \quad (26)$$

Curious

There is another solution to the ODE for the value of the aggregate endowment.

$$\frac{g(Y)\left(Y - \frac{\beta e^{a}\sigma^{2}}{2\rho}\right)}{\rho - \sigma^{2}} \tag{27}$$

It is not non-negative, but it does give a trading strategy which replicates the aggregate endowment payoffs. It satisfies the usual transversality conditions but these do not work here.

The Problem

- We have trouble when the aggregate endowment is not bounded away from zero.

- Two possible fixes
$$\delta(Y) = \epsilon + F(Y) \tag{28}$$
where $\epsilon > 0$ or $\exists \epsilon > 0$ such that $F(Y) \geq \epsilon$.

A Numerical Exercise

- We want to see what it takes to replicate some features of the data.
- We assume preferences have external habits with a deterministic habit.
- We assume that aggregate endowment is $e^{\mu_c t}(\exp(Y_t) + \epsilon)$
- We assume that the dividend is $e^{\mu_c t} \max\left[e^{Y_t} - \epsilon_D, 0\right]$
- $dY_t = -\kappa Y_t dt + \sigma dB_t$

Results

Table 1: Results of Calibrations

Panel A: Calibrated Parameters

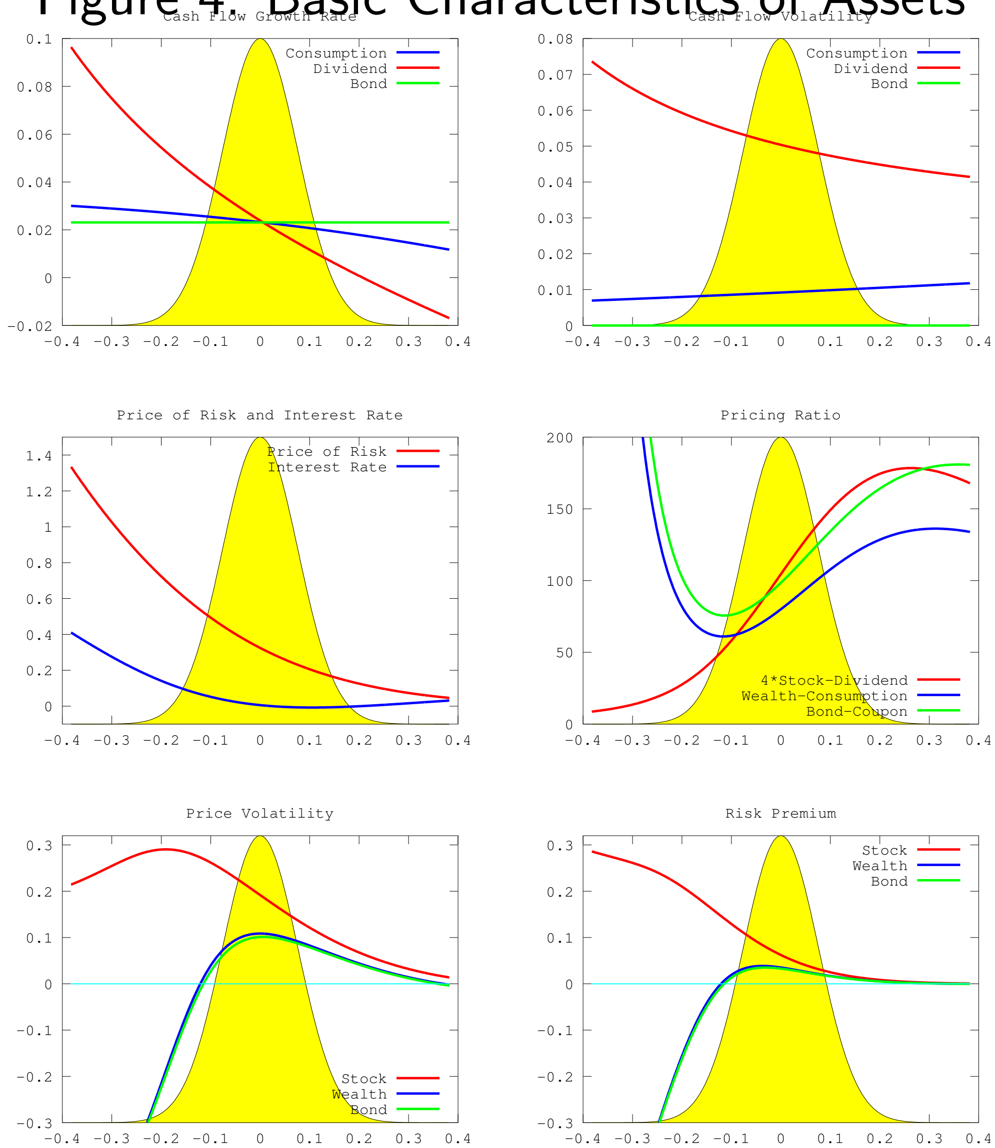
	$\tilde{a}$	ϕ	β	ρ	κ	σ	ϵ	ϵ_d
(1) Wealth	0.1736	0.0605	0.1210	0.0572	0.0770	0.0300	2.2634	0.4043
(2) Stock	0.1814	0.0631	0.1261	0.0606	0.0657	0.0300	2.2637	0.4038

Panel B: Unconditional Moments

	Data*	(1) Wealth	(2) Stock
Consumption Volatility	0.92%	**0.92%**	**0.92%**
Dividend Volatility	5.06%	**5.06%**	**5.06%**
Wealth-consumption ratio	83	**83**	**83**
Wealth Volatility	9.02%	**9.02%**	7.40%
Wealth Premium	2.38%	**2.38%**	2.03%
Stock-dividend Ratio	26	**26**	**26**
Stock Volatility	17.2%	19.1%	**17.2%**
Stock Premium	6.41%	7.09%	**6.41%**
SD of Stock Premium	3.31%	4.08%	3.76%
SD of Log Wealth-consumption Ratio	18.6%	19.7%	17.2%
SD of Log Stock-dividend Ratio	29.0%	36.8%	34.5%
Interest Rate	1.49%	1.53%	1.71%
SD of Interest Rate	-	2.74%	2.04%
Price of Risk	-	0.34	0.34

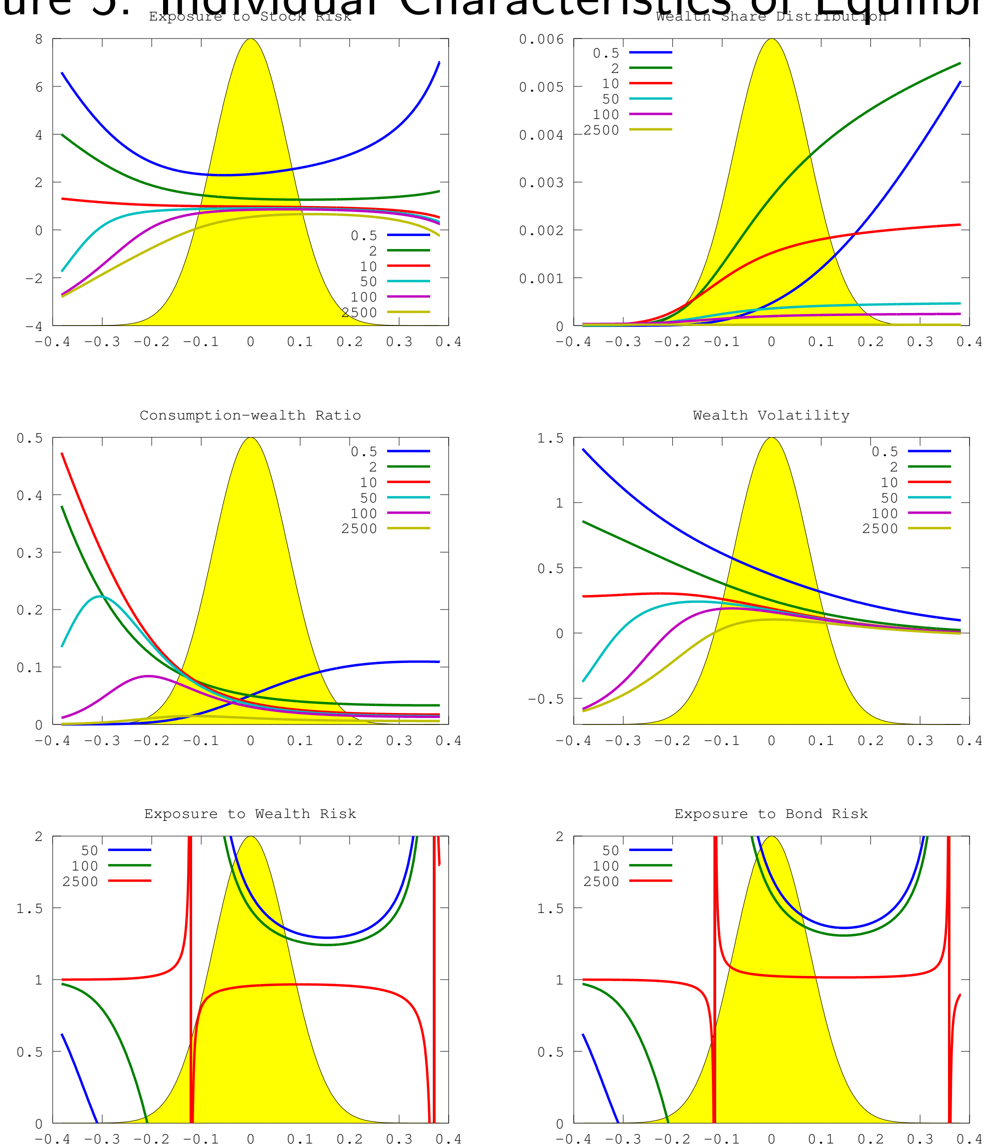
Asset Prices

Figure 4: Basic Characteristics of Assets



Individuals

Figure 5: Individual Characteristics of Equilibrium



Conclusion