Recursive Utility, Stochastic Volatility and Currency Risk Premium

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Introduction

- The exchange rate S_t, dollar priced per unit of foreign currency
- home short rate r_t and foreign short rate r_t^*
- Currency return $\frac{S_{t+1}e^{r_t^*}}{S_t}$
- The log-return, in log exchange rate, $s_t \equiv \ln S_t$ is: $s_{t+1} - s_t + r_t^*$
- The currency excess return is: $\rho_{t+1} = s_{t+1} - s_t + r_t^* - r_t$
- The risk premium is: $E_t(\rho_{t+1}) = E_t(s_{t+1} - s_t + r_t^* - r_t)$

The puzzle we want to explain:

Engel's observation (AER, 2016)

$cov(E_t(\rho_{t+1}), r_t^* - r_t) > 0$

$cov(\sum_{j=0}^{\infty} E_t(\rho_{t+j+1}), r_t^* - r_t) < 0$

$\Rightarrow$

$cov(E_t(\rho_{t+j+1}), r_t^* - r_t) < 0$ for large j

Does this forms a paradox?

The puzzle we want to explain:

Engel's observation (AER, 2016)

$$\rho_{t+j+1} = \alpha + \beta_j \times (r_t^* - r_t) + \varepsilon$$

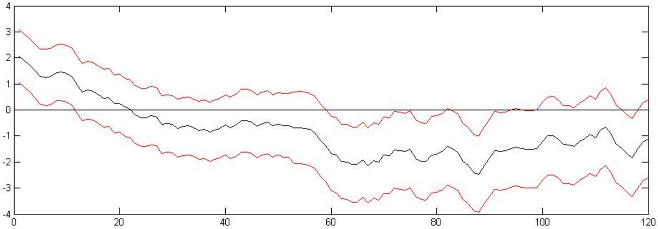


Figure: β_j of CAD/USD with j vary from 1 to 120 months.

Our Contribution

- We show that a fairly standard model with time varying risk premium can generate patterns documented in Engel (AER, 2016)
- Key features needed: recursive utility and stochastic volatility

Model: Utility

- A representative agent with a recursive utility

$$U_t^\rho = (1-\beta) C_t^\rho + \beta (E_t[U_{t+1}^\alpha])^{\frac{\rho}{\alpha}}$$

- β is the subjective discount coefficient
- $\gamma = 1 - \alpha$ is the risk-aversion coefficient
- $\theta = 1 - \rho$ is the inverse of elasticity of intertemporal substitution
- $\alpha = \rho$ is the case of expected utility with CRRA γ
- We study the special case of $\rho = 0$ for tractability

$$\ln U_t = (1-\beta) \ln C_t + \frac{\beta}{\alpha} \ln(E_t[U_{t+1}^\alpha])$$

Model: Stochastic Volatility

- We study the consumption growth has stochastic volatility

$$c_{t+1} - c_t = \mu + \lambda x_t + h x_t^2 + x_t \epsilon^c_{t+1}.$$

with $x_{t+1} = \varphi_x x_t + \sigma_x \epsilon^x_{t+1}$.

- Two shocks ϵ^c_{t+1} and ϵ^x_{t+1} are independent, μ, λ, h, φ and σ_x are all constant
- x_t is the conditional volatility of $c_{t+1} - c_t$ and has a mean-reversion coefficient of φ_x and conditional volatility of σ_x
- $\mu + \lambda x_t + h x_t^2$ is the conditional mean of consumption growth. h is not essential for quantitative conclusion but may be important for empirical calibration
- $\lambda > 0$ without loss of generality and is important

Closed-Form Value Function

- We can solve the value function in closed-form

$$V_t = C_t e^{a+bx_t+kx_t^2} \equiv C_t \Lambda_t,$$

with

$$b = \frac{\beta\lambda}{1 - \frac{\beta\varphi_x}{1-2\alpha w_{xx}\sigma_x^2}}$$

- b is positively proportional to λ
- and with

$$k = \frac{1}{4\alpha\sigma_x^2}\left(1 - \beta\varphi_x^2 + (\alpha^2 + 2\alpha h)\beta\sigma_x^2 - \sqrt{(1 + \beta\varphi_x^2 - (\alpha^2 + 2\alpha h)\beta\sigma_x^2)^2 - 4\beta\varphi_x^2}\right).$$

Closed-Form Value Function: Property of k

1. When $\alpha = 0$, utility reduces to expected utility with log utility

$$\frac{\partial k}{\partial \alpha} > 0$$

2. When $\sigma = 0$, the conditional volatility of consumption growth is non-stochastic

$$k > 0$$

Pricing Kernel

- The pricing kernel determines the short rate, the conditional mean of $\ln \pi_{t+1}$, and the risk premium of the model, given by the coefficients of the shock terms of $\ln \pi_{t+1}$
- In our case, $V_t = C_t \Lambda_t = C_t e^{a + bx_t + kx_t^2}$

$$\pi_{t+1} = \beta \underbrace{\frac{\mathrm{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]}{\mathrm{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\alpha}\right]}}_{e^{-r_t}} \times \underbrace{\frac{\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}}{\mathrm{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]} \frac{\Lambda_{t+1}^{\alpha}}{\mathrm{E}_t[\Lambda_{t+1}^{\alpha}]}}_{\xi_{t+1}}$$

The Short Rate

- The short rate is given by

$$r_t = -\ln\beta + \underbrace{(\mu + \lambda x_t + h x_t^2)}_{intertemporal\ \ substitution} - (\underbrace{\gamma}_{precautionary\ \ saving} - \underbrace{1/2}_{Jensen's\ \ effect})x_t^2,$$

- We can rewrite

$$r_t = -\ln\beta + \mu + \lambda x_t + (h - \gamma + 1/2)x_t^2.$$

- When $(h - \gamma + 1/2) < 0$ the short rate decreases in x_t^2.

The Risk Premium

- The shock terms in the pricing kernel determines the risk premiums.
- The risk premium for ϵ_{t+1}^{c} is determined by

$$\exp\left(-\frac{\gamma^2}{2}x_t^2 + \gamma x_t \epsilon_{t+1}^{c}\right)$$

- The market price for ϵ_{t+1}^{c} is γx_t

The Risk Premium

- The risk premium for ϵ_{t+1}^{x} is determined by

$$\frac{\exp\left(\alpha\sigma_x(b+2k\varphi_x x_t)\epsilon_{t+1}^{x}+\alpha k\sigma_x^2\epsilon_{t+1}^{x2}\right)}{(1-2\alpha k\sigma_x^2)^{-1/2}\exp\left(\frac{1}{2}\alpha^2\sigma_x^2(b+2k\varphi_x x_t)^2/(1-2\alpha k\sigma_x^2)\right)}$$

- The market price for ϵ_{t+1}^{x} is

$$\frac{(\alpha\sigma_x)(b+2k\varphi_x x_t)}{(1-2\alpha k\sigma_x^2)}$$

- This is zero if $\alpha=0$ (expected utility) or $\sigma_x=0$ (no stochastic volatility)

The Exchange Rate

- Suppose that home country and foreign country have pricing kernel π_{t+1} and π^*_{t+1} respectively
- The exchange rate S_{t+1} must satisfy, assuming complete markets, (Backus, Foresi and Telmer (2001, JF))

$$\frac{S_{t+1}}{S_t} = \pi^{-1}_{t+1}\pi^*_{t+1}$$

- In log terms,

$$s_{t+1} - s_t = -\ln \pi_{t+1} + \ln \pi^*_{t+1} \quad .$$

The Currency Risk Premium

- The currency risk premium if given by

$$E_t[\rho_{t+1}] = \frac{1}{2}\gamma^2 x_t^2 + \frac{\alpha^2\sigma_x^2(b+2k\varphi_x x_t)^2}{2(1-2\alpha k\sigma_x^2)} - \frac{1}{2}\gamma^2 x_t^{*2} - \frac{\alpha^2\sigma_x^2(b+2k\varphi_x x_t^*)^2}{2(1-2\alpha k\sigma_x^2)}$$

- We will assume that home consumption C_t and foreign consumption C_t^* are independent but have identical parameters.
- The correlateion between currency risk premium and the short rate differential $r_t - r_t^*$ is twice of the correlation between $\frac{1}{2}\gamma^2 x_t^2 + \frac{\alpha^2\sigma_x^2(b+2k\varphi_x x_t)^2}{2(1-2\alpha k\sigma_x^2)}$ and r_t, which we compute
- We will abuse the notation to write

$$\rho_t = \frac{1}{2}\gamma^2 x_t^2 + \frac{\alpha^2\sigma_x^2(b+2k\varphi_x x_t)^2}{2(1-2\alpha k\sigma_x^2)}$$

The Currency Risk Premium -cont.

- The stochastic part of the risk premium is

$$E_t[\rho_{t+1}] = \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}x_t + \left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)x_t^2.$$

- The first term is due to the linear term in consumption growth. The coefficient of the first term is positive if $k > 0$
- The second term is due to volatility. It is positive, which is dictated by Jensen's inequality.

Expected Currency Risk Premium

- The stochastic part of the expected currency risk premium is

$$E_t[\rho_{t+j+1}] = \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}E[x_{t+j}] + \left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)E[x_{t+j}^2]$$
$$= \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}\varphi_x^j x_t + \left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\varphi_x^{2j}x_t^2 \quad .$$

- The second term goes to 0 twice as fast as the first term
- The stochastic part of the short rate is

$$r_t = \lambda x_t + (h + \alpha - 1/2)x_t^2 \ .$$

Correlation

- The unconditional correlation between the expected risk premium and the short rate is

$$\mathrm{cov}[\mathrm{E}_t[\rho_{t+j+1}], r_t] = \lambda \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}\varphi_x^j \mathrm{Var}[x_t] + (h+\alpha-1/2)\left(\frac{\gamma^2}{2}+\frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\varphi_x^{2j}\mathrm{Var}[x_t^2].$$

and

$$\sum_{j=0}^{\infty}\mathrm{cov}[\mathrm{E}_t[\rho_{t+j+1}], r_t] = \lambda \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}\frac{\mathrm{Var}[x_t]}{1-\varphi_x} + (h+\alpha-1/2)\left(\frac{\gamma^2}{2}+\frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\frac{\mathrm{Var}[x_t^2]}{1-\varphi_x^2}.$$

Resolution of the Puzzles

- We need

$$\mathrm{cov}[\mathrm{E}_t[\rho_{t+1}], r_t] =$$
$$\lambda \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}\varphi_x E[x_t^2] + (h+\alpha-1/2)\left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\varphi_x^2 var[x_t^2] < 0$$

$$\frac{\alpha^2\lambda^2\beta k}{1-2\alpha\sigma^2 k-\beta\varphi_x} + (h+\alpha-1/2)\left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\frac{1+3\varphi^2}{1-\varphi^4} < 0$$

- Thus $(h+\alpha-1/2) < 0$ necessarily, and

$$\sum_{j=0}^{\infty}\mathrm{cov}[\mathrm{E}_t[\rho_{t+j+1}], r_t] =$$
$$\lambda \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}\frac{\varphi_x E[x_t^2]}{1-\varphi_x} + (h+\alpha-1/2)\left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\frac{\varphi_x^2 var[x_t^2]}{1-\varphi_x^2} > 0$$

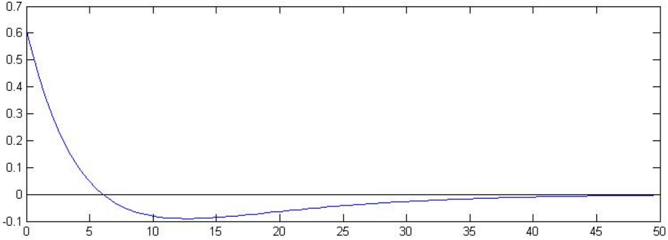
$$\frac{\alpha^2\lambda^2\beta k}{1-2\alpha\sigma^2 k-\beta\varphi_x} + (h+\alpha-1/2)\left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\frac{1+3\varphi^2}{(1+\varphi)(1-\varphi^4)} > 0$$

- thus φ_x needs to be large

The puzzle we want to explain:

Numerical example:

$$\beta_j = \frac{cov(E_t[\rho_{t+j+1}], r_t^* - r_t)}{var(r_t^* - r_t)}$$



Parameters Restrictions

- $\alpha\sigma_x \neq 0$: recursive utility and stochastic volatility are simultaneously required
- $h + \alpha < 1/2$: precautionary saving dominates intertemporal substitution and Jensen's effect
- $2h + \alpha > 0$: required for $k > 0$
- A sufficient condition proposed by Engel (2016), $\alpha < 0$ is not needed. For example, $\alpha = 1/4$ and $h = 0$ works.

Economic Intuition

- The short rate is

$$r_t = -\ln\beta + \underbrace{\mu + \lambda x_t + \Big(h}_{intertemporal\ \ substitution} + \underbrace{\alpha}_{precautionary\ \ saving} - \underbrace{1/2}_{Jensen's\ \ effect}\Big)x_t^2,$$

- The expected risk premium is

$$E_t[\rho_{t+j+1}] = \frac{2\alpha^2\sigma_x^2\varphi_x bk}{1-2\alpha k\sigma_x^2}\varphi_x^j x_t + \left(\frac{\gamma^2}{2} + \frac{2\alpha^2\sigma_x^2(\varphi_x k)^2}{1-2\alpha k\sigma_x^2}\right)\varphi_x^{2j}x_t^2 \quad .$$

- The x_t^2 term, which dominates when $|x_t|$ is large, leads to negative correlation.
- The x_t term, which dominates when $|x_t|$ is small, leads to positive correlation.
- In most studies have x_t^2 term

Economic Intuition -cont.

Prediction:

- When volatility shocks are small, r_t and $E_t[\rho_{t+j+1}]$ is positively correlated
- When volatility shocks are large, r_t and $E_t[\rho_{t+j+1}]$ is negatively correlated

Economic Intuition and Mechanism

- $\frac{S_{t+1}}{S_t} = \pi^*_{t+1}\pi^{-1}_{t+1}$
- By assuming consumption independency

$$E_t\left(\frac{S_{t+1}}{S_t}\right) = E_t[\pi^*_{t+1}]E_t[\pi^{-1}_{t+1}]$$

$$= e^{r^*_t}E_t[\pi^{-1}_{t+1}] = exp(\underbrace{r^*_t - r_t}_{interest\ \ parity} + \underbrace{\sigma^2_t}_{assuming\ \ conditional\ \ nomal})$$

- Thus ρ_{t+1} generally equal to square of market price of risk due to Jensen's inequality

$$E_t[\rho_{t+j+1}] \propto \sigma^2_t$$

$$r_t = -\ln E_t[\pi_{t+1}] = \mu_t - \sigma^2_t(\cdots)$$

- We can see that stochastic σ^2_t plays a crucial role in currency risk premium

Conclusions

- Recursive utility and stochastic volatility can lead to a resolution of currency risk premium puzzle raised by Engel, without behavioral models and/or sticky prices.
- We are looking at excess volatility generated by the model. The preliminary result indicates that we can generate volatility patterns of expected currency returns