Consumption and Bubbles

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This version: June 8, 2010

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Neoclassical asset pricing bubbles are often characterized as speculative phenomena in which investors pay more than the value of the asset's dividend stream in anticipation of receiving a profit by selling the asset later. For such bubbles, there cannot be a last date of trade, which suggests that an infinite number of trading opportunities is necessary to support a bubble. We show, however, that the number of dates at which investors consume is also an essential determinant for whether bubbles can exist. Our framework is a continuous-time model in which the number of trade dates is infinite but the number of consumption dates is flexible and can be chosen to be uniformly bounded, finite almost surely, or infinite. Within this framework, we show that market clearing, together with monotonically increasing preferences for consumption, limits the properties of bubbles and provides endogenous transversality conditions. In the special case of a uniformly bounded number of consumption dates, positive net supply assets cannot have asset pricing bubbles in an equilibrium.

1 Introduction

In neoclassical economics, an asset pricing bubble exists if the price of an asset exceeds the lowest cost of superreplicating its future dividends.[1] Some studies therefore characterize bubbles as speculative phenomena in which investors pay more than the value of the asset's dividend stream in anticipation of receiving a profit by selling the asset later. Models of bubbles typically use discrete-time and infinite-horizons (e.g. Santos and Woodford (1997)), but some use continuous-time and finite-horizons (e.g. Loewenstein and Willard (2000b)).[2] Both types require an unbounded number of dates for investors to engage in speculative trade to support a bubble; otherwise, given no arbitrage, a simple backwards induction argument will rule out such bubbles.

We study the number of dates at which investors can *consume* as a essential determinant for whether bubbles can exist. Our model assumes investors only consume at discrete dates but permits continuous trading over a finite horizon. This provides flexibility in choosing the number of consumption dates to be uniformly bounded across states, finite almost surely, or infinite. This is the first study to examine the effects on equilibrium asset pricing bubbles when the number of consumption dates has different finiteness properties than the number of trading dates in the asset market.

Our model allows for short-lived and long-lived investors who choose consumption-investment plans subject to lower bounds on negative wealth. We assume all investors prefer more consumption to less. Markets can be complete or incomplete. In this setting, the existence of an optimal consumption-investment strategy by itself cannot rule out asset pricing bubbles. Moreover, limited-scale arbitrage strategies are possible. These possibilities are not new to this framework and were addressed by Loewenstein and Willard (2000a,b). But what is different about our analysis is that we link the properties of asset pricing bubbles to consumption dates. As a result, we consider a subset of asset pricing bubbles and limited arbitrages that we call "local bubbles" and "local limited arbitrages," which are connected to a finite number of consumption dates. An asset has a local asset pricing bubble if there exists a cheaper alternate strategy which provides at least the same payouts at every time and in every state of nature for a fixed number of consumption dates. A local limited arbitrage is a trading strategy which requires no wealth and provides positive payoffs over a fixed number of consumption dates but is not feasible at all scales due to an investor's lower bound on wealth. Both local bubbles and local limited arbitrages are consistent

[1]We allow incomplete financial markets. As such, the dividends of some or all assets might not be exactly replicated by an alternate portfolio. An asset's dividends are superreplicated if there is a portfolio that has a dividend stream at least as great as the original asset's dividends.

[2]Bubbles in continuous-time models were introduced by Loewenstein and Willard (2000b). These bubbles can have uniformly bounded lifespans, and can have uniformly bounded values. Subsequently, several papers have studied properties of bubbles in partial equilibrium and general equilibrium settings. Cox and Hobson (2005) and Heston, Loewenstein, and Willard (2007) analyze bubbles in the context of option pricing. Jarrow, Protter, and Shimbo (2006, 2008) examine general semi-martingale models with bubbles. Huggonier (2009) shows how constraints on portfolio proportions invested in risky assets can introduce bubbles in models where investors are endowed with assets.

with an optimal consumption-investment strategy.

Our first main result shows that, *in equilibrium*, the prices of assets that contribute to aggregate financial wealth (those in positive net supply) cannot have local asset pricing bubbles. Moreover, in equilibrium, the investors' wealth constraints must satisfy a local transversality constraint that implies local limited arbitrage strategies cannot exist. As observed by Loewenstein and Willard (2000b), the existence of an optimal consumption-investment strategy by itself does not imply these results. Instead, we show that these results are implied by market clearing, together with the fact that investors who prefer more to less finance consumption at the lowest possible cost. More remarkably, these results hold whether or not the equilibrium value of the aggregate endowment is finite, unlike those in other studies of equilibrium asset pricing bubbles (e.g., Santos and Woodford (1997) and Loewenstein and Willard (2000b)).

An immediate corollary is that in models that have a uniformly bounded number of consumption dates across states of nature, there are no bubbles on the prices of assets in positive net supply and no limited arbitrages. This is true even though the number of trading dates is infinite, and it is true whether or not the value of the aggregate endowment is finite. This identifies the number of consumption dates as an essential determinant of whether bubbles can exist. The second corollary is that positive net supply assets whose payoffs occur on a uniformly bounded set of dates after which they cease trading (e.g., corporate bonds) cannot have bubbles. Again this is surprising since partial equilibrium conditions do not imply this nor does the work in Loewenstein and Willard (2000b).

The principal idea behind these results is that bubbles on the prices of the positive net supply assets must be accompanied by a sufficiently large accumulation of financial wealth. When investors prefer more to less, financial wealth represents the lowest cost of financing future net consumption. However, net consumption is automatically limited by market clearing because no investor's net consumption at a given date can exceed the cum-dividend price of the market portfolio and the amount of negative wealth permitted to all investors (i.e., an investor could consume net of endowments no more than the entire asset market plus what all investors could consume by running negative wealth to the limit). This limits the value of net consumption at each date. Over a fixed number of dates, the limit is sufficient to rule out the accumulation of wealth needed to support a bubble. When the number of consumption dates is uniformly bounded, market clearing thus rules out bubbles and limited arbitrages, and an assumption about the value of the aggregate endowment is unnecessary.

Over a nonuniformly bounded number of consumption dates, however, the bound may fail "at infinity." For economies in which the number of consumption dates is not uniformly bounded, local bubbles and local limited arbitrages still cannot exist, but bubbles and limited arbitrages that pay off at infinity can exist. Here, an assumption about equilibrium quantities is generally needed (we identify a few special cases that do not need an extra assumption). We use an assumption that appropriately bounds the investors' individual net consumption plans, and show that it rules out asset pricing bubbles regardless of the number of consumption dates. Moreover, we show that, given a suitable bound on aggregate wealth which rules out bubbles (e.g., a finite-valued aggregate endowment), then any long-lived investor's equilibrium wealth constraints must satisfy a transversality condition, and this rules out limited arbitrages even when the number of consumption dates can be infinite. In contrast to many infinite horizon equilibrium models, we do not assume these properties – they are derived as a necessary equilibrium condition given a suitable bound on aggregate wealth which rules out bubbles. This result suggests a deep connection between wealth constraints and bubbles through market clearing.

Section 5 presents an example of an equilibrium that not only highlights the role of the number of consumption dates but also has features that, to our knowledge, are the first of their kind. In the example, the final date of consumption is known, but no investor is sure whether he will consume at the final date or beforehand. This creates a model in which the number of consumption dates is almost surely finite but not uniformly bounded across paths. There is a bubble on the equilibrium price of the positive net supply asset. The bubble has a finite lifespan and is uniformly bounded; in fact, the asset's price itself is uniformly bounded. Both equilibrium consumption and consumption net of endowments are uniformly bounded. While our example uses a simple overlapping generations framework, equilibrium does not display indeterminacy typically encountered in discrete time infinite horizon overlapping generations models. Moreover, in a version with a long lived investor, not only is there an equilibrium bubble on a positive net supply asset, but the long lived investor's wealth constraint does not satisfy the usual transversality constraint.

Section 2 describes the model. Section 3 develops the preliminary results and explains bubbles and limited arbitrage. Section 4 contains our main results and shows how market clearing allows us to bound financial wealth. Section 5 contains our examples. Since there is a large literature on superreplication of cash flows, we provide extensions of these results to our setting in the Appendix. Additionally, the Appendix contains proofs of our results.

2 The Model

We begin our study of equilibrium asset pricing bubbles by describing the continuous-time model we use. The important features of the model are asset prices, investors' consumption-investment choice problems (including preferences and wealth constraints), and financial market equilibrium. We now describe these features.

2.1 Consumption Dates

Trade takes place on a time interval $[0, T]$, where T is finite and deterministic.[3] Uncertainty is represented by an underlying complete probability space with a probability

[3]Slight extensions would allow T to be a bounded stopping time or $T = \infty$.

measure P and a standard d-dimensional Brownian motion Z, and information arrival is described by the completed filtration $\{\mathcal{F}_t : t \in [0, T]\}$ generated by Z. This filtration describes the uncertainty in both asset prices and the timing of consumption dates, and as such represents the flow of information available to the investors. We adopt the convention that equalities and inequalities hold in the appropriate almost sure sense, and adopt the notation $E_t[x] \equiv E[x|\mathcal{F}_t]$.

The date T represents the last date of the economy. We allow the possibility that consumption prior to T might occur at dates that are potentially random. To describe these dates, we first let $t_0 = 0$ and $\{t_n : n = 1, \ldots, \infty\}$ denote a strictly increasing sequence of stopping times, and define the random number N to be $\sup\{n : t_n < T\}$ if a $t_n < T$ exists and 0 if not. If $N < \infty$ almost surely, then there is a last date of consumption and $N + 1$ is the number of consumption dates. In this case, we assume the last consumption date is time T, and accordingly redefine $t_{N+1} \equiv T$ so that the consumption dates along a given path ω are then described by the finite sequence $\{t_1(\omega), \ldots, t_{N(\omega)}(\omega), t_{N(\omega)+1}(\omega) = T\}$. We also allow the possibility that $N = \infty$ almost surely, in which case there is no last date of consumption, providing a sort of infinite-horizon model. In the latter case, we assume $\lim_{n\to\infty} t_n = T$. Our assumptions allow consumption at random dates and the possibility that the total number of such dates is unknown until time T.

2.2 Assets

There are $K + 1$ financial assets. The first asset is a locally riskless bond in zero net supply with a price process B that is strictly positive, predictable, and finite. We assume B has finite variation and satisfies for $t \leq t_n \wedge T$ for all n

$$B(t) = 1 + \int_0^t r(s)B(s)ds$$

for a progressively measurable locally riskless rate r. The remaining K assets are locally risky and pay dividends only on consumption dates. The cumulative dividends until time t for asset k are given by the right-continuous increasing process $D_k(0) + \sum_{0<t_j\leq t} D_k(t_j)$, where D_k is nonnegative and progressively measurable process. Let $D(t_j)$ denote the vector of dividends paid at $t_j \leq T$. Let S^{ex} denote the nonnegative vector process of the ex-dividend asset prices. Each S_k^{ex} satisfies

$$S_k^{\text{ex}}(t) = S_k^{\text{ex}}(0) + \int_0^t \mu_k(s)S_k^{\text{ex}}(s)ds + \sum_{j=1}^{d}\int_0^t \sigma_{kj}(s)S_k^{\text{ex}}(s)dZ(s) - \sum_{0<t_j\leq t} D_k(t_j)$$

for $t \leq t_n \wedge T$ and all n.[4] The processes μ_k and σ_{kj} are finite, progressively measurable, and satisfy the usual integrability conditions making the stochastic integrals well-defined (see Karatzas and Shreve (1988, Chapter 3.2)) on $[0, t_n \wedge T]$ for all n. Our analysis permits incomplete markets and locally redundant asset returns.

The net supplies of the risky assets, denoted by the $1 \times K$ row vector $\bar{\pi}_S$, are constant and nonnegative. (Modifications might allow for changing supplies of the assets.) Asset k will be in positive net supply if $\bar{\pi}_{S,k} > 0$.

2.3 Investors

Our model permits two types of investors: Those who potentially participate for the entire life of the economy, and those who participate for shorter periods but that arrive and depart on (possibly random) consumption dates, such as in overlapping generations models. We now describe the participation dates and preferences of these investors.

We assume there is a countable set of investors, and at most a finite number I of investors actively participate at any date $t \in [0, T]$. Investors are indexed by a positive integer i. For convenience, we assume investor 1 is present at time 0 and investor $i+1$ enters the economy no earlier than investor i. Let $\mathcal{I}_n$ denote the potentially random set of investors who may trade financial assets on the interval $[t_n, t_{n+1}]$. We also assume:

- Investors enter and exit (if at all) on consumption dates: investor i enters at time $\varsigma_i = \inf\{t_j | i \in \mathcal{I}_j\}$ and exits at time $\tau_i = \inf\{t_j \geq \varsigma_i | i \notin \mathcal{I}_j\}$ if such a time exists and $\tau_i = T$ otherwise.

- At any given consumption date t_n, each investor i knows whether he has entered the market and whether he is to exit: $\{i \in \mathcal{I}_n\}, \{i \notin \mathcal{I}_n\} \in \mathcal{F}_{t_n}$.

- An investor who has already exited does not reenter: for $t_n \geq \tau_i$, $\{i \notin \mathcal{I}_n\}$. When $N < \infty$, we set $\mathcal{I}_{N+1} = \emptyset$ which implies no investors trade after T.

- After entry, investor i may trade continuously: $\varsigma_i \leq t_n < \tau_i \Rightarrow i \in \mathcal{I}_n$. There is always at least one investor trading financial assets: $P(\mathcal{I}_n \neq \emptyset) = 1$ for $n \leq N$.

Investor i is *long-lived* if $P\{i \in \mathcal{I}_{n \wedge N}\} > 0$ for all n. Likewise, investor i is *short-lived* if there exists an $\bar{n} < N+1$ such that $\sup\{n - k | \tau_i = t_n, \varsigma_i = t_k\} \leq \bar{n}$; that is, short lived investors participate over at most a fixed number of dates. Our results

[4] This assumption implies each gains process $G_k(t) \equiv S_k^{\text{ex}}(t) + \sum_{0 < t_n \leq t} D_k(t_n)$ has almost surely continuous paths and does not have a singularly continuous with respect to Lebesgue measure component. We make this assumption to reduce notation. Given our assumed information structure, including discontinuities and singularly continuous components would be straightforward given a modest extension of our analysis. The assumption of the locally riskless asset in zero net supply is not restrictive if there is a portfolio that maintains strictly positive value at all consumption dates.

will be valid for models with only long-lived investors, models with only short-lived investors, and models with both types.[5]

Each investor i receives private endowments given by a nonnegative progressively measurable process e^i, where $e^i(t_n)$ is the endowment received at the consumption date t_n. Our analysis requires only that e^i is finite almost surely. Without loss of generality, we assume no investor receives endowments before entering the market or after exiting the market; i.e., $e^i(t_n) = 0$ for $t_n < \varsigma_i$ and $t_n > \tau_i$.

An investor's main object of choice is cumulative consumption. A cumulative consumption process is a nonnegative, nondecreasing, and progressively measurable process $C(t)$, with right continuous and left limited paths. By convention we set $C(0-) = 0$. Each investor i's preferences for consumption is an ordering $\succeq^i$ defined over cumulative consumption processes $C(t)$. If $C_1 \succeq^i C_2$, then the cumulative consumption process C_1 is weakly preferred to cumulative consumption process C_2. Cumulative consumption process C_1 is strictly preferred to cumulative consumption process C_2 written $C_1 \succ^i C_2$ if $C_1 \succeq^i C_2$ and $C_2 \not\succeq^i C_1$. Our later assumptions about preferences and equilibrium will later ensure that in equilibrium the investors' cumulative consumption processes increase only at consumption dates, but for now it is useful to allow "off-equilibrium" cumulative consumption plans that may increase at any date.

Because our model allows short-lived and long-lived investors, our assumptions about preferences are important for consistency between consumption and the receipt of endowments and dividends. Specific equilibrium models would include assumptions about this. (Our examples in Section 5 provide such specific assumptions.) To avoid limiting our general analysis, we only assume the following general properties for our model. Each property could be derived from more primitive assumptions.

Assumption 2.1. *Let $\underline{0}$ denote the "zero cumulative consumption process" ($C(t) \equiv 0$), and let $E(t)$ denote the cumulative consumption process that corresponds to consuming the aggregate endowment:*

$$E(t) = \bar{\pi}_S D(0) + \sum_i e^i(0) + \sum_{0<t_j\leq t} \left(\bar{\pi}_S D(t_j) + \sum_i e^i(t_j) \right).$$

We assume the following:

1. *Investor i can be no worse off if we add more consumption at all dates to an existing cumulative consumption process: Given any cumulative consumption process C and any other cumulative consumption process $\hat{C}$ then $C + \hat{C} \succeq^i C$.*

2. *Investor i strictly prefers consuming the aggregate endowment to not consuming at all: $E \succ^i \underline{0}$.*

[5] We exclude investors who enter subsequent to time 0 and participate with positive probability at all future dates. Our results do not depend on this exclusion. Instead, the exclusion greatly reduces bookkeeping complexity.

3. Preferences are strictly monotone in the following sense: There exists a time $\hat{\varsigma}_i$ with $\varsigma_i \le \hat{\varsigma}_i \le \tau_i$ such that given a cumulative consumption process C with $E \succeq^i C \succ^i \underline{0}$ and $\hat{C}$ with $\hat{C}(t) = 0$ for $t < \hat{\varsigma}_i$

$$\hat{C}(t) = \hat{C}(\hat{\varsigma}_i) + \sum_{\hat{\varsigma}_i < t_n \le t} \hat{c}(t_n)$$

and $P\{\hat{C}(t_k \wedge \tau_i) > 0\} > 0$ for some k, then $C + \hat{C} \succ^i C$.

Part 1 of Assumption 2.1 is a standard weak monotonicity assumption and would be implied by free disposal of consumption. It implies $C \succeq^i \underline{0}$. Part 2 ensures that preferences are at least minimally consistent with the timing of endowments and are nonsatiated in equilibrium. Part 3 defines directions of strict improvement for the investors' preferences. If the investor prefers a candidate optimum to not consuming at all, then adding consumption after some date will improve upon the candidate optimum. This would be true if all investors strictly prefer more to less at each consumption date, but our assumption is slightly weaker. Part 3 also implies if the investor is short-lived and exits at $\tau_i < T$, then he strictly prefers more to less consumption at this date. For a long-lived investor, it implies that at any consumption date prior to the end of the economy there exists some future consumption date where he strictly prefers more to less.

2.4 Budget Constraints

Each investor i present at the inception of the economy ($i \in \mathcal{I}_0$) additionally receives an endowment of $\pi_B^i(0)$ shares of the bond and $1 \times K$ row vector $\pi_S^i(0)$ shares of the risky assets, giving initial financial wealth $\pi_B^i(0)B(0) + \pi_S^i(0)S^{\text{ex}}(0)$. Because we assume the net supply of securities is fixed, investors who arrive subsequent to the inception of the economy receive no endowments of securities.

Investors may consume from their endowments and their gains from trade. Each investor i may trade continuously when present in the economy (on (ς_i, τ_i)). Denote a trading strategy by the progressively measurable process $\pi^i = (\pi_B^i, \pi_S^i)$, where $\pi_B^i(t)$ represents shares of the bond and the K-dimensional row vector $\pi_S^i(t) = [\pi_{S1}^i(t), \ldots, \pi_{Sk}^i(t), \ldots, \pi_{SK}^i(t)]$ represents shares of the risky assets at time t.

Given these features, every investor i's financial wealth process W^i must satisfy the budget equation for $t \le t_n \wedge T$ and all n

$$\begin{aligned} W^i(t) = \pi_S^i(t)S^{\text{ex}}(t) + \pi_B^i(t)B(t) = W^i(0) + \int_{(0,t]} \pi_S^i(s)dS^{\text{ex}}(s) \\ + \sum_{0<t_j\le t} \pi_S^i(t_j)D(t_j) + \int_0^t \pi_B^i(s)dB(s) + \sum_{0<t_j\le t} e^i(t_j) - (C(t) - C(0)) \end{aligned} \quad (2.1)$$

where $W^i(0) = \pi_B^i(0)B(0) + \pi_S^i(0)S^{\text{ex}}(0) + \pi_S^i(0)D(0) - C(0) + e^i(0)$. This of course implies for any stopping time χ and $t \in [\chi, t_n \wedge T]$ for any n

$$W^i(t) = \pi^i_S(t)S^{\mathrm{ex}}(t) + \pi^i_B(t)B(t) = W^i(\chi) + \int_{(\chi,t]} \pi^i_S(s)dS^{\mathrm{ex}}(s) + \sum_{\chi<t_j\leq t} \pi^i_S(t_j)D(t_j) + \int_\chi^t \pi^i_B(s)dB(s) + \sum_{\chi<t_j\leq t} e^i(t_j) - (C(t) - C(\chi)) \quad (2.2)$$

The budget equation requires consumption to be financed through trading gains and endowments. We assume π^i makes the integrals describing portfolio gains in (2.1) well-defined (see Karatzas and Shreve (1988, Chapter 3.2)) on $[0, t_n \wedge T]$ for all n.

Although we describe the individuals' wealth processes on $[0, t]$, our framework allows overlapping generations. Since the net supply of securities is fixed, we would have $\pi^i_B(0) = 0$, $\pi^i_S(0) = 0$ and $W^i(t) = 0$ prior to the entry of a given individual into the economy. Then according to 2.1, $W^i(\varsigma_i) = e^i(\varsigma_i) - C(\varsigma_i)$. If the economy has a bounded number, $\bar{I}$, of potential investors we set $W^i \equiv 0$ for all $i > \bar{I}$.

2.5 Optimal Consumption and Investment Choice

We now present the investors' consumption and investment choice problems. Constraints on negative wealth are important in continuous-time models because they make "doubling strategies" infeasible at some scale (Harrison and Pliska, 1981; Dybvig and Huang, 1988). Such constraints are typically either "endogenous" or "exogenous," the meanings of which are described below.

We use the notation a^i for a process that describes lower bound on the wealth process of investor i. Here is our main assumption about each a^i.

Assumption 2.2. *For investor i, a lower bound a^i on wealth is pathwise nonpositive ($P\big(\forall t \in [\varsigma_i, t_n \wedge T]\ a^i(t) \leq 0\big) = 1$ for all n) and is the value of a portfolio which may make payments through time and does not allow investor i to exit in debt. That is, (i) for $\varsigma_i \leq t \leq t_n \wedge T$ and all n*

$$a^i(t) = \alpha^i_S(t)S^{ex}(t) + \alpha^i_B(t)B(t) = a^i(\varsigma_i) + \int_{(\varsigma_i,t]} \alpha^i_S(s)dS^{ex}(s) + \sum_{\varsigma_i<t_j\leq t} \alpha^i_S(t_j)D(t_j) + \int_{\varsigma_i}^t \alpha^i_B(s)dB(s) - \sum_{\varsigma_i<t_j\leq t} g^i(t_j) \quad (2.3)$$

for a progressively measurable portfolio $\alpha^i = (\alpha^i_B, \alpha^i_S)$ and a progressively measurable process $g^i \leq 0$, and (ii) if $\tau_i = t_n$ for a finite n, then $a^i(t) = 0$ for $t \geq \tau_i$. We assume $a^i(\varsigma_i) \in \mathcal{F}_{\varsigma_i}$ and is finite almost surely. We also assume the portfolio α satisfies conditions ensuring the integrals in (2.3) are well-defined on $[0, t_n \wedge T]$ for all n.

Because a^i is nonpositive, it is always feasible for investor i to immediately liquidate his securities positions (if any), consume the proceeds, and not to trade at any future date. Furthermore, since a^i is described by a portfolio strategy that allows interim payments (reflected by the term g^i) would allow, for example, the situation where investor i can borrow up to the lower bound of the present values of his future endowments when markets are incomplete (as in Santos and Woodford (1997)). Although a^i does not allow investor i to exit in debt, we have *not* assumed $a^i(T) = 0$ when $N = \infty$.

We consider two types of wealth constraints. One type of wealth constraint is the *exogenous* constraint, described by the following consumption-investment choice problem.

Choice Problem 2.1 (Exogenous Wealth Constraints)**.** *Given securities endowments $\pi^i_B(0)$ and $\pi^i_S(0)$, endowments e_i, and a fixed constraint on negative wealth a^i satisfying Assumption 2.2, choose a cumulative consumption process C^i that satisfies $C^i \succeq^i C$ among the set of cumulative consumption processes C for which there is a corresponding portfolio π for which the wealth process W satisfies the budget equation* (2.1)*, $W(t) = 0$ for $t < \varsigma^i$, and the lower bound on wealth*

$$P\big((\forall t \in [\varsigma_i, t_n \wedge T])\ W(t) \geq a^i(t)\big) = 1 \quad \forall\ n. \tag{2.4}$$

A solution is the pair (C^i, π^i): the optimal consumption plan C^i and the corresponding portfolio π^i that finances it.

Constraint (2.4) applies to every portfolio that investor i might choose. The interpretation is a monitoring agency or trading partners determine an investor's creditworthiness and limit negative wealth by monitoring wealth as it evolves (Dybvig and Huang, 1988; Magill and Quinzii, 1994; Loewenstein and Willard, 2000a). Dybvig and Huang (1988) and Loewenstein and Willard (2000a) consider the special cases of nonnegative wealth ($a \equiv 0$) and negative wealth bounded in units of the bond ($W(t) \geq a(t) \equiv -\gamma B(t)$ for fixed $\gamma > 0$). Other cases include $a(t) \equiv -\gamma S_k(t)$ and $a(t) \equiv -\gamma(\hat{\pi}_B B(t) + \hat{\pi}_S S(t))$ for fixed $\gamma > 0$ $\hat{\pi}_B \geq 0$ and $\hat{\pi}_S \geq 0$, which would limit negative wealth by a fixed portfolio or the market portfolio as a numeraire.

In contrast, Delbaen and Schachermayer (1994, 1995, 1997b,a) allow investor i to choose a lower bound for negative wealth simultaneously with a portfolio. As Delbaen and Schachermayer's results also concern bubbles, we include their *endogenous* constraints in our study, as defined in the following choice problem.

Choice Problem 2.2 (Endogenous Wealth Constraints)**.** *Given securities endowments $\pi^i_B(0)$ and $\pi^i_S(0)$, endowments e_i, and a collection $\mathcal{A}^i$ of lower bounds on wealth, choose a cumulative consumption process C^i that satisfies $C^i \succeq^i C$ among the set of cumulative consumption processes C for which there is a corresponding portfolio π with a wealth process W that satisfies the budget equation* (2.1)*, $W(t) = 0$ for $t < \varsigma^i$, and the lower bound on wealth*

$$(\exists a^i \in \mathcal{A}^i) \quad P\big((\forall t \in [\varsigma_i, t_n \wedge T])\ W(t) \geq a^i(t)\big) = 1 \quad \forall\ n. \tag{2.5}$$

A solution is the triplet (C^i, π^i, a^i): the optimal consumption plan C^i and the corresponding portfolio π^i that finances it given the lower bound $a^i \in \mathcal{A}^i$.

We now assume endogenous wealth constraints can be scaled arbitrarily to distinguish them economically from exogenous constraints.

Assumption 2.3. *In Problem 2.1* $\mathcal{A}^i \neq \{0\}$. *Each* $a \in \mathcal{A}^i$ *satisfies Assumption 2.2 and*

$$a_1, a_2 \in \mathcal{A}^i \;\Rightarrow\; a_1 + a_2 \in \mathcal{A}^i. \tag{2.6}$$

The enforcement of endogenous constraints relies on an investor's perception of a limit on negative wealth (Magill and Quinzii, 1994), but does not fix a specific bound. Every feasible strategy can be scaled by integer amounts in Problem 2.2 since $na \in \mathcal{A}^i$ if $a \in \mathcal{A}^i$. Delbaen and Schachermayer (1994, 1995, 1997b,a) study the special case $\mathcal{A}^i = \{a(t) = -\gamma B(t) : \gamma \in \Re_+\}$, which uses the locally riskless bond as the numeraire. It is important to note that a solution to Problem 2.2 includes both a specific trading strategy and a specific lower bound on wealth.

2.6 Equilibrium

We ultimately identify some necessary properties of equilibrium prices given exogenous and endogenous bounds on negative wealth. Here we define equilibrium.

Definition 2.1. *An* ***equilibrium*** *consists of asset prices* (B, S) *satisfying the assumptions of Section 2, cumulative consumption processes* $\{C^i : i = 1, \ldots\}$ *with* $C^i \succ^i \underline{0}\ \forall i$ *and portfolios* $\{\pi^i : i = 1, \ldots\}$ *such that:*

1. *Given the asset prices, every investor i's consumption and portfolio solves Problem 2.1 or Problem 2.2 given Assumptions 2.2, 2.3, and 2.1,*
2. *The asset markets clear:* $\forall n$

$$\bar{\pi}_B = \sum_i \pi^i_B(t) = 0 \quad \textit{and} \quad \bar{\pi}_S = \sum_i \pi^i_S(t) \qquad t \in [0, t_n], \tag{2.7}$$

and

3. *The consumption market clears: for all* $t \leq t_n \wedge T$, $\forall n$

$$\sum_i C^i(t) = \sum_{0 \leq t_j \leq t} \left(\bar{\pi}_S D(t_j) + \sum_i e^i(t_j) \right).$$

We study only the properties of prices necessary for an equilibrium to exist. While we do not study sufficient conditions for an equilibrium, our conclusions about equilibrium asset pricing and bubbles are valid for any equilibrium of a specific model satisfying our assumptions.

Remark 2.1. *Within a given equilibrium, each investor i's constraint on negative wealth* a^i *can be regarded as fixed. This is automatic for exogenous constraints. Even for endogenous constraints, the existence of an equilibrium would require every investor to choose an optimal portfolio and a corresponding* $a^i \in \mathcal{A}^i$ *so that it is feasible given the bound on negative wealth* (2.5).

3 Bubbles and Arbitrage Given Optimal Choice

We now provide assumptions which ensure our model is consistent with optimal choice. Some of our results in this section are similar to those in previous work. However, the scope of our model is more general in terms of assumptions about market completeness, the types of wealth constraints, and the form the constraints may take. This generality will later be important for our main analysis and requires different techniques, so we present complete extensions of the results that we use.

3.1 State Prices

Our study of asset pricing bubbles will ultimately compare asset prices to their fundamental values. Fundamental value inherently involves a notion of "state prices." Here we identify necessary properties of state prices given optimal portfolio choice in our model, allowing for incomplete markets and locally redundant assets.

Recalling the notation in Section 2.2, let $\sigma(t)$ be the $K \times d$ asset volatility matrix $(\sigma_{kj}(t))_{k=1,\dots K;j=1,\dots,d}$ of the risky assets, let $\mu(t)$ be the column vector of $(\mu_k(t))_{k=1,\dots,K}$ of local expected returns, and let $S(t)$ be the column vector of the risky asset prices $(S_k(t))_{i=1,\dots,d}$. The matrix $\sigma(t)$ might not be invertible (incomplete markets) and might have rank less than K (locally redundant assets).

We first state two assumptions to ensure that investor choice problems at least are consistent with a candidate equilibrium.[6]

Assumption 3.1. *There exists a progressively measurable process $\tilde{\theta}$ such that*

$$\mu(t) - r(t)\mathbf{1_K} = \sigma(t)\tilde{\theta}(t), \tag{3.8}$$

Lebesgue$\times P$-almost everywhere on $[0, t_n \wedge T]$ for all n.

The process $\tilde{\theta}$ is often called the "local price of risk." The local price of risk reflects the condition that two portfolios with the same volatility must have the same drift, or else there would be an arbitrage that maintains nonnegative wealth. See Karatzas and Shreve (1998, Theorem 1.4.2) for the construction of this strategy. Such an arbitrage would be inconsistent with a solution for either choice problem above; hence, a local price of risk is necessary for optimal choice.

Given $\tilde{\theta}$ of Assumption 3.1, let $\theta(t)$ be its orthogonal projection onto the range of $\sigma'(t)$ for all $t \in [0, T]$, where prime denotes transpose. Karatzas and Shreve (1998, Lemma 1.4.4) show θ is progressively measurable. Define the stopping time τ by $\tau = \inf\big\{t \in [0,T] \mid \int_0^t ||\theta(s)||^2 ds = \infty\big\}$ where $\tau = \infty$ if such a t does not exist. We assume $P(\tau = 0) = 0$. Define

$$\rho^0(t) = \frac{\exp\Big(-\int_0^t \theta'(s)dZ(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\Big)}{B(t)} \tag{3.9}$$

[6]Earlier versions of this paper showed that these assumptions are really necessary conditions for existence of an optimal strategy under additional mild assumptions about the monotonicity of preferences. The corresponding lengthy proofs were distracting to our main points, so this version simply labels the conditions as assumptions.

on $\{t < \tau\}$ and $\rho^0(t) = 0$ on $\{t \geq \tau\}$.

The process ρ^0 can be thought of as a state price density. In discrete-state models, zero state prices imply zero-cost arbitrage strategies inconsistent with optimal choice (Dybvig and Ross, 1987). But Loewenstein and Willard (2000a) show in continuous-time models that zero state prices might imply only "approximate arbitrages" that require vanishingly positive initial investment, maintain nonnegative wealth, and generate ever larger payoffs on $\{\rho^0(t_n) = 0\}$. These approximate arbitrages and the assumptions about preferences that would rule them out distract from our main points, so we make the following assumption.

Assumption 3.2. *$\rho^0(t_n \wedge T) > 0$ for all n.*

Remark 3.1. *When $N < \infty$ so that the last date of the consumption is at time T, our assumption says $\rho^0(T) > 0$. However, when $N = \infty$, our assumption neither implies that $\lim_{t \to T} \rho^0(t)$ exists nor does that it is strictly positive. Our assumption also does not imply the existence of an equivalent martingale measure.*

Together, Assumptions 3.1 and 3.2 rule out strategies which are obvious "arbitrages:" strategies that start with no wealth, maintain nonnegative wealth and payoff strictly positive cumulative consumption with positive probability. As discussed in Dybvig and Huang (1988) and Loewenstein and Willard (2000a), such strategies would be inconsistent with any of our investors having optimal solutions since, given a candidate optimum, an investor can feasibly add such a strategy at any scale.

Proposition 3.1. *Given Assumptions 3.1 and 3.2, any strategy which satisfies 2.1 with $e^i \equiv 0$, $\pi_B^i(0) \equiv 0$,$\pi_S^i(0) \equiv 0$ and maintains nonnegative wealth must have $C(t) = 0$ for all $t \in [0, t_n \wedge T]$ and all n.*

Proof. See Appendix A.2. □

This proposition does not say strategies which start with no endowments, receive no endowments, and obey a wealth constraint described in Assumption 2.2 must have $C(t) = 0$ for all $t \in [0, t_n \wedge T]$ and all n. Such a strategy may not be feasible at any scale when added to a candidate optimum as we explain in the next sections.

3.2 Bubbles and Limited Arbitrage

The main point of this section is that asset pricing bubbles and limited-scale arbitrage opportunities are consistent with optimal choice in the present model. Thus they may be regarded as "consistent" with partial equilibrium. Section 4 will show, however, they are inconsistent with the market clearing required by an equilibrium.

The next subsection defines and discusses asset pricing bubbles; the subsequent subsection will define and discuss limited arbitrage opportunities.

3.2.1 Asset Pricing Bubbles

The Law of One Price says two portfolios having the same payouts have the same price. A violation of the Law of One Price is often associated with an asset pricing bubbles, which we now define.

Definition 3.1. *Asset k has an* ***asset pricing bubble*** *if its price exceeds the lowest cost of a portfolio that 1) satisfies the budget equation* (2.1) *with cumulative consumption which is at least as high as the asset k's cumulative dividend : $C(t_n) \geq \sum_{j=1}^{n\wedge N+1} D_k(t_j)$ for all n and 2) maintains nonnegative wealth $W(t) \geq 0$.*

The portfolio in Definition 3.1 will be said to superreplicate the payoffs of asset k. Our definition is consistent with that in Santos and Woodford (1997). A bubble might seem to be inconsistent with optimal choice since it creates an arbitrage opportunity. However, arbitraging a bubble involves short selling the higher-cost asset and buying the lower-cost superreplicating portfolio. The feasibility of this strategy depends on the nature of the bubble, the dates investors can trade, and on the lower bounds on negative wealth, as we describe in this section and the next.

We also study a subset of asset pricing bubbles that have "finite" lifetimes, as first identified in Loewenstein and Willard (2000b). We will call them "local asset pricing bubbles" (see Jarrow et al. (2006, 2008) for a related idea).

Definition 3.2. *Asset k has a* ***local asset pricing bubble*** *if, given a fixed n, the price exceeds the lowest cost of a portfolio that 1) satisfies the budget equation* (2.1) *with cumulative consumption which is at least as high as the asset k's cumulative dividend until time $t_n \wedge T$: $C(t_m) \geq \sum_{j=1}^{m\wedge N+1} D_k(t_j)$ for all $m \leq n$, 2) has time $t_n \wedge T$ value which exceeds the asset value : $W(t_n \wedge T) \geq S_k(t_n \wedge T)$ and 3) maintains nonnegative wealth $W(t) \geq 0$.*

A local asset pricing bubble is inconsistent with the existence of an equivalent martingale measure on $[0, t_n]$ for all n. As such, discrete-time infinite-horizon models (as in Santos and Woodford (1997)) cannot have local bubbles. However, as discussed in Loewenstein and Willard (2000a,b), the existence of an equivalent martingale measure is not necessary for existence of an optimum in a continuous time model when investors face wealth constraints. Loewenstein and Willard (2000a,b) and Heston et al. (2007) provide examples of local bubbles.

Bubbles inherently involve the lowest superreplication cost. We now link superreplication costs to state prices for incomplete markets. Let $\mathcal{V}$ denote the set of progressively measurable d-dimensional processes ν with $\sigma(t)\nu'(t) = 0$, Lebesgue $\otimes P$ a.s., and $\int_0^T ||\nu(s)||^2 ds < \infty$, P-a.s. For a given $\nu \in \mathcal{V}$, define

$$\rho^\nu(t) = \frac{\exp\left(-\int_0^t \big(\theta'(s) + \nu(s)\big) dZ(s) - \frac{1}{2}\int_0^t ||\theta(s) + \nu'(s)||^2 ds\right)}{B(t)} \tag{3.10}$$

Every ρ^ν is nonnegative and continuous. In this context, financial markets are complete if $\mathcal{V} = \{0\}$. Given Assumption 3.2 and $\int_0^T ||\nu(s)||^2 ds < \infty$ and Revuz and

Yor (1994, Exercise IV.3.25) imply $P(\rho^\nu(t_n \wedge T) > 0) = 1$ for all n and $\forall \nu \in \mathcal{V}$ and we can link the ρ^ν's to the replicating costs of dividends. Karatzas and Shreve (1998) provides many results on costs of superreplicating a given sequence of payouts. Section A.1 in the Appendix provides extensions of these results to our setting.

The possibility of bubbles in a partial equilibrium reflects the observation that

$$\rho^\nu(t) S_k^{\text{ex}}(t) + \sum_{j=1}^{N+1} \rho^\nu(t_j) D_k(t_j) 1_{\{t_j \le t\}},$$

and $\rho^\nu(t)B(t)$ are only nonnegative local martingales (supermartingales) for each $\nu \in \mathcal{V}$.[7] They are not necessarily martingales. Therefore, $\forall \nu \in \mathcal{V}$,

$$S_k^{\text{ex}}(0) \ge E\left[\sum_{j=1}^{n\wedge N+1} \rho^\nu(t_j) D_k(t_j) + \rho^\nu(t_n \wedge T) S_k^{\text{ex}}(t_n \wedge T)\right] \tag{3.11}$$

and

$$B(0) \ge E\left[\rho^\nu(t_n \wedge T) B(t_n \wedge T)\right] \tag{3.12}$$

An inequality in (3.12) or (3.11) is strict if and only if the corresponding process is not a martingale. Given a strict inequality for all choices of ν, there is a local asset pricing bubble.

Proposition 3.2. *Given Assumption 3.2, if the inequality*

$$S_k^{ex}(0) > \sup_{\nu \in \mathcal{V}} E\left[\sum_{j=1}^{n\wedge N+1} \rho^\nu(t_j) D_k(t_j) + \rho^\nu(t_n \wedge T) S_k^{ex}(t_n \wedge T)\right] \tag{3.13}$$

holds, then there is a trading strategy that satisfies (2.1) *with initial wealth* $w < S_k^{ex}(0)$, $e^i \equiv 0$, *generates cumulative consumption* $C(t) \ge \sum_{0<t_j\le t} D_k(t_j)$ *for* $t \le t_n \wedge T$ *and a terminal payoff* $W(t_n \wedge T) \ge S_k^{ex}(t_n \wedge T)$, *and maintains pathwise nonnegative wealth.*

Proof. See Appendix A.2. □

Proposition 3.2 identifies when asset k has a local asset pricing bubble. The righthand side of inequality (3.13) is the lowest cost of superreplicating asset k's dividends and time $t_n \wedge T$ payoff given pathwise nonnegative wealth. An analogous result holds for the locally riskless bond price B when

$$B(0) > \sup_{\nu \in \mathcal{V}} E\left[\rho^\nu(t_n \wedge T) B(t_n \wedge T)\right]. \tag{3.14}$$

[7]The local martingale property follows from Ito's Lemma. A process X is a local martingale if there is an increasing sequence of stopping times $\{\tau_n\}$ such that $\lim_{n\to\infty} \tau_n = T$ almost surely and each stopped process $X(t \wedge \tau_n)$ is a martingale. A nonnegative local martingale is a supermartingale (Karatzas and Shreve, 1988, Exercise 1.5.19). Strict local martingales are "explosive" on small probability sets in that they satisfy both $E[\max_{t\in[0,T]} \rho^\nu(t) S_k(t)] = \infty$ and $P(\max_{t\in[0,T]} \rho^\nu(t) S_k(t) \ge \lambda) \le S_k(0)/\lambda$ (Protter (1992, Theorem I.47) and Revuz and Yor (1994, Theorem II.1.7)).

In this case, the bond would have a local asset pricing bubble.

At this point, readers new to the idea of finite-lived bubbles in continuous-time models might want examples. However, the literature already contains numerous examples with complete discussion of how they are consistent with optimal portfolio choice. We refer the reader to Loewenstein and Willard (2000a,b) and Heston et al. (2007) for explicit closed-form examples of bubbles – some for well-known models of asset prices – that are consistent with optimal portfolio choice and strictly monotone preferences. We also present an (albeit more complex) example of a bubble in Section 5.

One main focus of this paper is identifying conditions for which there are no local asset pricing bubbles.

Proposition 3.3. *If there is a* $\nu^* \in \mathcal{V}$ *such that* $\rho^{\nu^*}(t)S_k^{ex}(t)+\sum_{j=1}^{N+1}\rho^{\nu^*}(t_j)D_k(t_j)1_{\{t_j\le t\}}$ *is a martingale on* $[0, t_n \wedge T]$ *for all* n*, then*

$$(\forall n) \quad S_k^{ex}(0) = E\left[\sum_{j=1}^{n\wedge N+1}\rho^{\nu^*}(t_j)D_k(t_j) + \rho^{\nu^*}(t_n\wedge T)S_k^{ex}(t_n\wedge T)\right] \tag{3.15}$$

and asset k *does not have a local asset pricing bubble.*

Proof. See Proposition A.1. □

If a ν^* with the properties in Proposition 3.3 exists, then the stock price represents the lowest cost of superreplicating asset k's dividends with a portfolio that maintains nonnegative wealth. The existence of such a ν^* is *not* implied by the existence of an optimal consumption-investment strategy. We will show, however, that the existence will be implied by existence of an equilibrium in Section 4.

Local asset pricing bubbles are a subset of asset pricing bubbles. To see this, apply Monotone Convergence and Fatou's Lemma to (3.15), to get

$$\begin{aligned} S_k^{\text{ex}}(0) &\ge E\left[\sum_{j=1}^{N+1}\rho^{\nu^*}(t_j)D_k(t_j) + \liminf_{n\to\infty}\rho^{\nu^*}(t_n\wedge T)S_k^{\text{ex}}(t_n\wedge T)\right] \\ &\ge E\left[\sum_{j=1}^{N+1}\rho^{\nu^*}(t_j)D_k(t_j)\right]. \end{aligned} \tag{3.16}$$

It is possible that an asset does not have a local asset pricing bubble yet has an asset pricing bubble. An asset pricing bubble would be related to attaching a price to payoffs not in the consumption set. For example in an infinite horizon model, an asset pricing bubble might be thought of as attaching value to payoffs at infinity. Given this, we should be careful to note that it will be difficult to provide conditions to rule out asset pricing bubbles on zero net supply assets since they can effectively promise payoffs outside the consumption set. Our next result identifies a property of an asset price with an asset pricing bubble.

Proposition 3.4. *The inequality*

$$S_k^{ex}(0) > \sup_{\nu \in \mathcal{V}} E\left[\sum_{j=1}^{N+1} \rho^\nu(t_j) D_k(t_j)\right] \tag{3.17}$$

implies there is a trading strategy that satisfies (2.1) *with initial wealth* $w < S_k^{ex}(0)$, $e^i \equiv 0$, *with cumulative consumption* $C(t) \geq \sum_{0<t_j \leq t} D_k(t_j)$ *for* $t \leq t_n \wedge T$ *for all* n, *and maintains pathwise nonnegative wealth.*

Proof. The proof follows directly from Proposition A.2: take $a \equiv 0$, $\gamma(t_j) = D_k(t_j)$, and $w = \sup_{\nu \in \mathcal{V}} E\left[\sum_{j=1}^{N+1} \rho^\nu(t_j) D_k(t_j)\right]$. □

The righthand side of inequality (3.17) is the lowest cost of superreplicating asset k's dividends given pathwise nonnegative wealth, thus if inequality (3.17) holds, asset k has an asset pricing bubble. For example, when $N = \infty$ in our model, the bond would always have an asset pricing bubble because it pays no dividends. This should cause no concern, however, since it is in zero net supply. If $N < \infty$ one might interpret the terminal value equal to a liquidating dividend and if $B(0) > \sup_{\nu \in \mathcal{V}} E[\rho^\nu(T) B(T)]$ there would exist a portfolio which superreplicates the terminal value of the bond which requires initial wealth less than $B(0)$ and maintains nonnegative wealth.

3.2.2 Wealth Constraints and Limited Arbitrage

The preceding subsection identifies properties of asset prices that lead to asset pricing bubbles. We now study the investors' lower bounds on wealth. These lower bounds are described by portfolios of assets, and the assets in these portfolios may contain bubbles. Even if the assets have no bubbles, these portfolios might follow suicide strategies (see, e.g., Harrison and Pliska (1981)) that "throw away" value through time. If this is the case, then the payouts $-g^i$ in the lower bound can be superreplicated at a cost lower than $-a^i(\varsigma_i)$, the initial wealth required by the portfolio that defines the negative wealth constraint.

This produces "limited arbitrages" as first identified by Loewenstein and Willard (2000a,b). These are like unlimited arbitrages in that they require no investment and provide a positive probability of future consumption (with zero probability of negative consumption). Unlimited arbitrages, however, maintain nonnegative wealth and therefore are not limited by lower bounds on wealth. Limited arbitrages differ by *requiring* some amount of negative wealth. Constraints on negative wealth therefore limit investors' positions in these strategies because at large enough scale the investor would violate the constraint.

In this section, we connect the properties of wealth constraints and state prices to identify conditions under which limited arbitrages can or cannot exist. These conditions vary slightly between short-lived and long-lived investors.

We begin with the case for long-lived investors because it is more direct.

Proposition 3.5. *Let a^i be a given constraint on negative wealth satisfying Assumption 2.2 with $\varsigma_i = 0$. The process $-\rho^\nu(t)a^i(t) - \sum_{j=1}^{N+1} \rho^\nu(t_j)g^i(t_j)1_{\{\varsigma_i < t_j \le t\}}$ is a nonnegative local martingale and hence a supermartingale. Moreover, the inequality*

$$-a^i(0) > \sup_{\nu \in \mathcal{V}} E\left[\sum_{j=1}^{N+1} -\rho^\nu(t_j)g^i(t_j)\right] \tag{3.18}$$

holds if and only if there is a portfolio trading strategy and cumulative consumption process such that the corresponding wealth process satisfies the budget equation (2.1) *with no initial wealth* ($(W(0) \le 0)$ *and* $e^i = 0$)*, provides positive consumption* ($C(t) \ge 0$ *and* $C \not\equiv 0$)*, and honors the constraint on negative wealth* ($P((\forall t \in [0, t_n \wedge T])\, W(t) \ge a^i(t)) = 1\ \forall n$)*. This strategy, however, must risk negative wealth* ($P((\exists t \in [0,T))\, W(t) < 0) > 0$)*.*

Proof. See Appendix A.2. □

The absence of the limited arbitrages for long-lived investors in Proposition 3.5 then requires the equality

$$-a^i(0) = \sup_{\nu \in \mathcal{V}} E\left[\sum_{j=1}^{N+1} -\rho^\nu(t_j)g^i(t_j)\right]. \tag{3.19}$$

In Section 4, we provide conditions for equilibrium which produce equality (3.19) and provide new restrictions on asset prices and wealth constraints.

The corresponding result for short-lived investors is more complicated because they enter and exit at different times and because we only assume $a^i(\varsigma_i)$ is finite almost surely. As such, we will often present our results on an $\mathcal{F}_{\varsigma_i}$ measurable subset such that $\sup_{\nu \in \mathcal{V}} E[-\rho^\nu(\varsigma_i)a^i(\varsigma_i)1_{\{A\}}] < \infty$. This will allow us to make statements conditional on the subset A on which the conditional expectations will be defined. For short-lived investors, we have the following analog to Proposition 3.5.

Proposition 3.6. *Let a^i be a given constraint on negative wealth satisfying Assumption 2.2. Then on any $A \in \mathcal{F}_{\varsigma_i} \bigcap \{t_n > \varsigma_i\}$ with $\sup_{\nu \in \mathcal{V}} E[-\rho^\nu(\varsigma_i)a^i(\varsigma_i)1_{\{A\}}] < \infty$, the process $-\rho^\nu(t)a^i(t) - \sum_{j=1}^{N+1} \rho^\nu(t_j)g^i(t_j)1_{\{\varsigma_i < t_j \le t\}}$ is a nonnegative local martingale and hence a supermartingale for all $\nu \in \mathcal{V}$. Moreover, the inequality*

$$-a^i(\varsigma_i)1_{\{A\}} > \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_{\varsigma_i}\left[\left(\sum_{j=1}^{n \wedge N+1} -\rho^\nu(t_j)g^i(t_j)1_{\{t_j > \varsigma_i\}} - \rho^\nu(t_n \wedge \tau_i)a(t_n \wedge \tau_i)\right)1_{\{A\}}\right]}{\rho^\nu(\varsigma_i)} \tag{3.20}$$

holds if and only if there is a portfolio trading strategy and cumulative consumption process such that the corresponding wealth process satisfies the budget equation (2.1) *with no initial wealth* ($W(0) = 0$)*,* $e^i = 0$)*, provides positive consumption $P(C(t_n \wedge \tau) \ge 0) = 1$, provides nonnegative wealth by some date* ($(\exists n)\ P(W(t_n \wedge \tau_i)1_{\{A\}} > 0) = 1$)*, and honors the constraint on negative wealth* ($P((\forall t \in [\varsigma_i, t_n \wedge \tau_i])\, W(t) \ge a^i(t)) =$

1).[8] *This strategy, however, must risk negative wealth* $(P((\exists t \in [\varsigma_i, t_n \wedge \tau_i])\, W(t) < 0) > 0)$.

Proof. See Appendix A.2. □

The strategy in Proposition 3.6 differs from that in Proposition 3.5 in that the former guarantees nonnegative financial wealth by some consumption date. The former therefore represents a limited arbitrage that can obtained over a finite number of consumption dates; in this manner, it resembles a short-lived bubble. The absence of such strategies in Proposition 3.6 therefore requires the equality

$$-a^i(\varsigma_i)1_{\{A\}} = \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_{\varsigma_i}\left[\left(\sum_{j=1}^{n\wedge N+1} -\rho^\nu(t_j)g^i(t_j)1_{\{\varsigma_i<t_j\}} - \rho^\nu(t_n\wedge\tau_i)a(t_n\wedge\tau_i)\right)1_{\{A\}}\right]}{\rho^\nu(\varsigma_i)} \tag{3.21}$$

for all A with the properties listed in Proposition 3.6. In Section 4, we show that the existence of equilibrium will produce equality (3.21) and provide new restrictions on asset prices and wealth constraints.

We remark that for the endogenous constraints in Problem 2.2, we get equalities (3.19) and (3.21) automatically. This is because the existence of an optimum rules out strategies like those in Propositions 3.5 and 3.6 because the endogenous constraints can be arbitrarily scaled (Assumption 2.3). Thus equalities (3.19) and (3.21) would hold for all $a^i \in \mathcal{A}^i$.

We now turn to restrictions implied by the existence of an equilibrium.

4 Equilibrium

Having demonstrated that bubbles and limited arbitrage are potential "partial equilibrium" properties consistent with optimal portfolio choice and constraints on negative wealth, we now show that equilibrium provides additional restrictions. Short-lived bubbles and short-lived limited arbitrages are automatically inconsistent with the market clearing of an equilibrium. Long-lived bubbles and long-lived limited arbitrages are not automatically inconsistent with equilibrium, but can be ruled out if equilibrium net consumption satisfies certain properties. Our results identify equilibrium consumption as a new important ingredient in the formation of bubbles.

Remark 4.1. *From Definition 2.1 we observe that in equilibrium, each investor's cumulative consumption must satisfy*

$$C^i(t) = \sum_{0\le t_j\le t} c^i(t_j)$$

for the nonnegative adapted process $c^i(t_j) \equiv \Delta C^i(t_j)$. *For this section it is more convenient to work with the process* c^i.

[8] Essential supremum describes the least upper bound for a set of random variables. See footnote 15 for the definition.

We remind the reader of our main assumptions.

Assumption 4.1. *Assumption 3.2 holds; i.e.,* $\rho^0(t_n \wedge T) > 0$ *for all* n. *Lower bounds on wealth are described by the nonpositive values of self-financing portfolios as in Assumption 2.2. For endogenous constraints, set* $\mathcal{A}^i$ *has the scaling property in Assumption 2.3. Every investor chooses an optimal portfolio for Problem 2.1 or Problem 2.2. Furthermore, investor preferences satisfy Assumption 2.1.*

Given Assumption 4.1, our main results boil down to the observation that bubbles on positive net supply assets are not possible if every investor i's financial wealth is appropriately bounded. Our first main result is that this bound is *automatic* in an equilibrium over a uniformly bounded number of periods. This will rule out short-lived bubbles and short-lived limited arbitrages. When the number of periods is not uniformly bounded, however, the bound is not automatic and must be shown to hold in equilibrium. Our second main result shows that assuming this property of equilibrium net consumption will rule out all asset pricing bubbles. In this manner, our main results tie the formation of bubbles to the properties of the consumption set, rather than just the number of trading opportunities.

Our first main result highlights the automatic nature of the bounds for short-lived asset pricing bubbles and short-lived limited arbitrages.

Theorem 4.1. *Assume an equilibrium exists. Then there exists a* $\nu^* \in \mathcal{V}$ *such that*

1. *The equilibrium prices of the positive net supply assets do not have local asset pricing bubbles; that is, if* $\bar{\pi}_{S,k} > 0$,

$$S_k^{ex}(0) = E\left[\sum_{j=1}^{n\wedge N+1} \rho^{\nu^*}(t_j) D_k(t_j) + \rho^{\nu^*}(t_n \wedge T) S_k^{ex}(t_n \wedge T)\right], \tag{4.22}$$

   and inequality (3.11) *holds with equality.*

2. *There are no limited arbitrage opportunities over any given fixed number of consumption dates for both exogenous and endogenous constraints; that is, for every investor* i *and* $A \in \mathcal{F}_{\varsigma_i}$ *with* $\sup_{\nu\in\mathcal{V}} E[-\rho^\nu(\varsigma_i) a^i(\varsigma_i) 1_{\{A\}}] < \infty$,

$$-a^i(\varsigma_i)1_{\{A\}} = \frac{E_{\varsigma_i}\left[\sum_{j=1}^{n\wedge N+1} -\rho^{\nu^*}(t_j) g^i(t_j) 1_{\{t_j > \varsigma_i\}} - \rho^{\nu^*}(t_n \wedge \tau_i) a(t_n \wedge \tau_i)\right]}{\rho^{\nu^*}(\varsigma_i)} 1_{\{A\}} \tag{4.23}$$

   and inequality (3.21) *holds with equality for each* a^i. *For endogenous constraints, equality* (4.23) *additionally holds for every* $a^i \in \mathcal{A}^i$.

Proof. See Section A.3.1. □

Theorem 4.1 shows that local asset pricing bubbles that would affect equilibrium aggregate financial wealth and short-lived limited arbitrages are simply incompatible with equilibrium. Returning to our theme, we illustrate how market clearing limits wealth accumulation. For illustration we consider times up to time t_1. Notice for each investor $m \in \mathcal{I}_0$, $W^m(t) \geq a^m(t)$ implies $W^m(t_1-) \geq a^m(t_1-)$ and from this it follows $c^m(t_1) - e^m(t_1) + W^m(t_1) - (a^m(t_1) + g^m(t_1)) \geq 0$. Therefore,

$$\begin{aligned} c^m(t_1) - e^m(t_1) + W^m(t_1) - (a^m(t_1) + g^m(t_1)) \\ \leq \sum_{i \in I_0} \left(c^i(t_1) - e^i(t_1) + W^i(t_1) - (a^i(t_1) + g^i(t_1)\right) \\ = \bar{\pi}_S S^{\text{ex}}(t_1) + \bar{\pi}_S D(t_1) - \sum_{i \in \mathcal{I}_0} \left(a^i(t_1) + g^i(t_1) \right), \end{aligned} \tag{4.24}$$

where the equality follows from market clearing. The quantities in the last expression are nonnegative payouts of a portfolio with finite cost; namely, those of the market portfolio and the sum of the portfolios that describe the investors' lower bounds on negative wealth. Our monotonicity assumptions also imply that each investor's wealth must satisfy on $t \leq t_1$ (see Proposition A.3)

$$W^m(t) - a^m(t) = \text{essup}_{\nu \in \mathcal{V}} E_t \left[\frac{\rho^\nu(t_1)}{\rho^\nu(t)} \left(c^m(t_1) - e^m(t_1) + W^m(t_1) - (a^m(t_1) + g^m(t_1)) \right) \right] \tag{4.25}$$

The appearance of terms related to the wealth constraint in Equation (4.25) may seem unusual given more familiar results in Karatzas and Shreve (1998); however they reflect the ability of investor i to exploit any limited arbitrage if one should exist.[9] Equations (4.25) and (4.24) then imply

$$W^m(t) - a^m(t) \leq \text{essup}_{\nu \in \mathcal{V}} E_t \left[\frac{\rho^\nu(t_1)}{\rho^\nu(t)} \left(\bar{\pi}_S S^{\text{ex}}(t_1) + \bar{\pi}_S D(t_1) - \sum_{i \in \mathcal{I}_0} \left(a^i(t_1) + g^i(t_1) \right) \right) \right] \tag{4.26}$$

The right hand side of the above expression represents the value of a portfolio which superreplicates the payoff $\bar{\pi}_S S^{\text{ex}}(t_1) + \bar{\pi}_S D(t_1) - \sum_{i \in \mathcal{I}_0} (a^i(t_1) + g^i(t_1))$ and maintains nonnegative wealth. If the above expression did not hold the investor could improve by following the strategy

$$a^m(t) + \text{essup}_{\nu \in \mathcal{V}} E_t \left[\frac{\rho^\nu(t_1)}{\rho^\nu(t)} \left(\bar{\pi}_S S^{\text{ex}}(t_1) + \bar{\pi}_S D(t_1) - \sum_{i \in \mathcal{I}_0} \left(a^i(t_1) + g^i(t_1) \right) \right) \right] \geq a^m(t) \tag{4.27}$$

and investing the surplus in a portfolio which generates positive consumption and maintains nonnegative wealth. The payoff to this strategy would be higher than

$$\bar{\pi}_S S^{\text{ex}}(t_1) + \bar{\pi}_S D(t_1) - \sum_{i \in \mathcal{I}_0} \left(a^i(t_1) + g^i(t_1) \right) + a^m(t_1) + g^m(t_1) \tag{4.28}$$

But this would be inconsistent with equilibrium since it would imply other investors would have to lose more than their lower bounds on wealth permit. Summing over

[9] See Loewenstein and Willard (2000a,b) for a similar idea in a less general setting.

all $i \in \mathcal{I}_0$ and recall we assume at most I investors are present at time 0, we have

$$\bar{\pi}_S S^{\text{ex}}(t) - \sum_{i \in \mathcal{I}_0} a^i(t_1) \leq I \times \text{essup}_{\nu \in \mathcal{V}} E_t \left[\frac{\rho^\nu(t_1)}{\rho^\nu(t)} \left(\bar{\pi}_S S^{\text{ex}}(t_1) + \bar{\pi}_S D(t_1) - \sum_{i \in \mathcal{I}_0} \left(a^i(t_1) + g^i(t_1) \right) \right) \right] \tag{4.29}$$

These bounds and Lemma A.4 now imply there is no local bubble on positive net supply assets or limited arbitrage over the interval $[0, t_1]$. For an arbitrary interval $[t_j, t_{j+1}]$, the same analysis goes through if the economy is composed only of long lived investors; some care is required to deal with short lived investors who arrive after t_0 since we only assume their endowments and lower bounds on wealth are finite almost surely. We address this in Section A.3.1, but the proof largely relies on this type of analysis.

Before moving to the implications of Theorem 4.1, first note that the derivation of the above bounds does not depend on the characteristics of investors arriving at time t_1. As a result, if we want to assume new investors arrive with endowments of securities, the main ideas of Theorem 4.1 can still be derived although the statement of the theorem would be more complex due to the need to keep track of the asset supplies through time. Secondly, although we have assumed that investors choose the portfolios to be $\mathcal{F}_t$ progressively measurable, the above bounds still hold even if some investor potentially is allowed access to information represented by a finer filtration. In this sense, asymmetric information will not change our main conclusions.

We state a corollary.

Corollary 4.1. *Assume an equilibrium exists. Given the ν^* in Theorem 4.1, for any positive net supply asset k for which $S^{ex}(t) = 0$ for $t \geq t_{n+1}$ for some n, then asset k has no asset pricing bubble since*

$$S_k^{ex}(0) = E \left[\sum_{j=1}^{N+1} \rho^{\nu^*}(t_j) D_k(t_j) \right]. \tag{4.30}$$

Thus an asset that pays dividends only over a uniformly bounded number of periods and ceases trade cannot have an asset pricing bubble. Moreover, if there is a uniformly bounded number of consumption dates (i.e., there exists $\bar{n}$ with $N+1 \leq \bar{n}$), there are no asset pricing bubbles on any positive net supply asset.

Proof. Follows directly from Theorem 4.1. □

Corollary 4.1 illustrates the power of Theorem 4.1 for ruling out local, or "short-lived" asset pricing bubbles. Under our assumptions, equilibrium rules out local asset pricing bubbles on assets in positive net supply. So assets that cease to trade after a fixed number of dates, such as corporate debt, cannot have asset pricing bubbles.

More importantly, Corollary 4.1 gives us our main result: In an economy that has a strict finite lifetime – with the number of consumption dates uniformly bounded – no positive net supply assets can have asset pricing bubbles, whether local or not, and investors' wealth constraints cannot permit any limited arbitrage. In contrast to previous studies this finding does not depend on the value of the aggregate endowment. A potentially unbounded number of consumption dates is a necessary condition for a positive net supply asset to have a bubble.

For infinite-lived economies in which assets may have infinite lifetimes, such as equity or fiat money, Theorem 4.1 cannot rule out the possibility of a bubble. While bounds over a finite number of periods is automatic, summing the costs of such bounds over an infinite number of periods might not provide a finite cost. Hence, in general, an extra condition is needed. We will use the following condition. It is satisfied under other assumptions sometimes used in the literature, as we will explain.

Condition 4.1. *There exists a nonnegative progressively measurable process γ such that every investor i's net consumption satisfies $c^i(t_n) - e^i(t_n) \leq \gamma(t_n)$, where*

$$\sup_{\nu \in \mathcal{V}} E\left[\sum_{n=1}^{N+1} \rho^{\nu}(t_n)\gamma(t_n)\right] < \infty. \tag{4.31}$$

The process γ in Condition 4.1 must be the same for every investor.

Proposition 4.1. *If Condition 4.1 is satisfied in equilibrium, then under the same assumptions as Theorem 4.1 there are no asset pricing bubbles on the equilibrium price of any asset in positive net supply. That is, there exists a ν^* such that*

$$S_k^{ex}(0) = E\left[\sum_{n=1}^{N+1} \rho^{\nu^*}(t_n) D_k(t_n)\right]. \tag{4.32}$$

and inequality (3.17) *holds with equality. Additionally, for every long lived investor, equilibrium requires wealth constraints to satisfy the transversality condition*

$$-a^i(t) = \text{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} -\rho^{\nu}(t_j) g^i(t_j) 1_{\{t_j > t\}}\right]}{\rho^{\nu}(t)} = -\frac{E_t\left[\sum_{j=1}^{N+1} \rho^{\nu^*}(t_j) g^i(t_j) 1_{\{t_j > t\}}\right]}{\rho^{\nu^*}(t)}.$$

Proof. See Section A.3.2. □

Thus under Condition 4.1 there are no asset pricing bubbles on positive net supply assets. In addition, for long lived investors, there are no bubbles on the portfolios which limit negative wealth. In the case where the number of consumption dates cannot be uniformly bounded across states existence of equilibrium rules out doubling strategies, Ponzi schemes, or shorting portfolios with bubbles not just on $[0, t_n]$ as in Theorem 4.1 but on the entire life of the economy even if the economy does not have trade at T as would be the case when $N = \infty$. This result is important since it says under the same conditions which rule out bubbles, equilibrium also requires long lived investors' wealth constraints to obey a transversality condition. This result thus displays an important connection between wealth constraints and bubbles through market clearing.

To see how Condition 4.1 implies bounds on wealth accumulation, we sketch the main ideas of the proof of Proposition 4.1. The condition in Equation (4.31) implies there exists a portfolio strategy which maintains nonnegative wealth and superreplicates the payoff stream $\gamma(t_j)$. For all investors, in particular short lived investors, financial wealth must satisfy

$$W^i(t) \leq \text{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\gamma(t_j)1_{\{t_j>t\}}\right]}{\rho^\nu(t)} \tag{4.33}$$

because, if this inequality did not hold, given Assumption 2.1 part 3, the investor would derive higher utility by appropriately switching to the superreplicating strategy for γ (see Proposition A.2) which maintains nonnegative wealth and consuming more than $\gamma + e^i$ and, consequently, more than the equilibrium consumption C^i.

Additionally, each long lived investor who prefers more to less finances consumption net of endowments at its lowest possible cost, so each long lived investor i's equilibrium financial wealth must satisfy

$$W^i(t) - a^i(t) \leq \text{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=n+1}^{N+1} \rho^\nu(t_j)\big(\gamma(t_j) - g^i(t_j)\big)\right]}{\rho^\nu(t)} \quad \text{on } [t_n, t_{n+1})$$

because, if this inequality did not hold, the investor would derive higher utility by appropriately switching to the superreplicating strategy for γ which exploits any limited arbitrage (see Proposition A.2) and consuming more than $\gamma + e^i$ and, consequently, more than the equilibrium consumption C^i. Second, using this for long lived investors, (A.22) for short lived investors, plus market clearing for the assets implies aggregate financial wealth must be bounded:

$$\begin{aligned}
&\bar{\pi}_S S^{\text{ex}}(t) - \sum_{i,\text{long lived}} a^i(t) = \sum_i W^i(t) - \sum_{i,\text{long lived}} a^i(t) \\
&\leq I \times \text{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=n+1}^{N+1} \rho^\nu(t_j)\big(\gamma(t_j) - \sum_{i,\text{long lived}} g^i(t_j)\big)\right]}{\rho^\nu(t)} \quad \text{on } [t_n, t_{n+1}). \qquad (4.34)
\end{aligned}$$

Showing that this rules out asset pricing bubbles and implies transversality constraints on long lived investors now requires some technical results we take up in Section A.3.2.

A necessary condition typically associated with the existence of bubbles on positive net supply assets is the need for frequent trade (so frequent, in fact, to be unbounded). Theorem 4.1 and Proposition 4.1 add a new necessary condition of frequent consumption when investors prefer more to less. For a model with a uniformly bounded number of consumption dates there cannot be asset pricing bubbles on positive net supply assets. When we have a potentially unbounded number of consumption dates, we would want to verify Condition 4.1 holds to rule out asset pricing bubbles on positive net supply assets. This same condition then immediately implies investor equilibrium wealth constraints must also satisfy transversality constraints both for endogenous and exogenous constraints. Section 5 shows the importance of the uniform bound presenting an equilibrium bubble given an almost surely finite, but not uniformly bounded, number of consumption dates. In addition, in a variant of the model with a long lived investor, the long lived investor's wealth constraint does not satisfy a transversality condition.

The following result helps to explains how our result differs from what is known in the literature in discrete time infinite horizon models.

Corollary 4.2. *In equilibrium, Condition 4.1 is satisfied when the present value of the aggregate endowment is finite; i.e., when*

$$\sup_{\nu \in \mathcal{V}} E\left[\sum_{n=1}^{N+1} \rho^{\nu}(t_n) e(t_n)\right] < \infty. \tag{4.35}$$

Regardless of the number of consumption dates, an appropriate choice of γ given this assumption is

$$\gamma(t_n) = e(t_n) + \bar{\pi}_S D(t_n).$$

Inequality (4.35) is known to be satisfied in an "asset economy" in which the endowments of the consumption good are identically zero (i.e., $e^i \equiv 0$ for every investor i) or in which endowments are bounded by some multiple of the assets' aggregate dividends. This is true regardless of the number of consumption dates, and choosing γ to be proportional to $\bar{\pi}_S D(t_n)$ would be appropriate for Condition 4.1. See Santos and Woodford (1997) further discussion along these lines for discrete-time models.

As mentioned before, a necessary condition often associated with bubbles on positive net supply assets is frequent trade in the asset markets. The polar opposite is an economy with no trade in the asset market. Our next proposition indicates for economies with no trade in the asset market there are no bubbles on positive net supply assets. The conditions listed in the proposition are only meaningful for an economy composed only of long lived investors.

Corollary 4.3. *In any equilibrium with no trade in the asset market, or if $c^i(t_n) - e^i(t_n) \geq 0$ for all i and all n, Condition 4.1 is satisfied. Regardless of the number of consumption dates, an appropriate choice of γ in this case is*

$$\gamma(t_n) = \pi^{max} D(t_n).$$

where $\pi^{\max}$ is $[\pi_1^{\max}, \ldots, \pi_K^{\max}]$ with $\pi_k^{\max} = \max_{i \in \mathcal{I}_0} \pi_{S,k}^i$.

Proof. Given no trade in the asset market, each individual's net consumption must satisfy $c^i(t_n) - e^i(t_n) = \pi_S^i D(t_n) \leq \pi^{\max} D(t_n)$. When $c^i(t_n) - e^i(t_n) \geq 0$, then $c^i(t_n) - e^i(t_n) \leq \sum_i (c^i(t_n) - e^i(t_n)) = \bar{\pi}_S D(t_n) \leq \pi^{\max} D(t_n)$. □

Loosely speaking, in economies populated with investors who have a high propensity to consume endowments at every date, it will be harder to generate bubbles. Thus economies which generate bubbles should be associated with investors who have a low propensity to consume endowments, perhaps due to precautionary savings, in a certain sense. This is a feature present in our examples in Section 5.

In general, however, whether or not the value of the aggregate endowment is finite, there is no trade, or individuals optimally consume more than their endowment is determined endogenously within the equilibrium. For an economy with a uniformly bounded number of consumption dates, we have shown there are no bubbles on positive net supply assets regardless of whether these conditions hold. When the number of consumption dates is not uniformly bounded, the assumption is often made that the present value is finite (see, e.g., Santos and Woodford (1997) and Loewenstein and Willard (2000b)).

5 Example

5.1 Positive Net Supply Asset with an Equilibrium Bubble

We now present an example of an equilibrium in which the price of a positive net supply asset has a bubble. The purpose of the example is to illustrate the importance of the number of consumption dates. In the example, the number of consumption dates is finite almost surely, but is not uniformly bounded across states. Otherwise, the critical economic quantities are uniformly bounded: The asset's price is uniformly bounded; therefore, so is the bubble on its price. The bubble has a finite lifespan. Consumption and private endowments are also uniformly bounded. All investors prefer more consumption to less, and choose the lowest cost portfolio to finance their net consumption. The lack of a uniform bound on the number of consumption dates causes Condition 4.1 to be violated, so consumption net of endowments cannot be superreplicated by a finite-cost portfolio in this example.[10]

The financial market consists of two assets, one net unit of a "stock" that pays only a liquidating dividend of $D(T) = 3/2$ at date T and zero net units of a locally riskless bond. Their prices are S and B, and continuous trade is permitted over the deterministic time interval $[0, T]$. Uncertainty is described by two standard and independent Brownian motion processes Z_1 and Z_2.

The consumption dates constructed from a sequence of stopping times we now define. Define an exponential local martingale η by

$$\eta(t) = \exp\left(-\frac{1}{2}\int_0^t \psi^2(s)ds + \int_0^t \psi(s)dZ_1(s)\right),$$

where ψ is some given deterministic process having the properties

$$(\forall t \in [0, T)) \quad \int_0^t \psi^2(s)ds < \infty \quad \text{and} \quad \int_0^T \psi^2(t)dt = \infty$$

[10]Some of the mathematical constructs for our example are similar to Delbaen and Schachermayer (1998).

almost surely. Note η is independent of Z_2. The Novikov condition ensures $E[\eta(t)] = 1$ for all $t \in [0,T)$; however, $\eta(T) = 0$ almost surely and η is not a martingale. Define the stopping time τ by

$$\tau = \inf \left\{ t \in [0,T] \,:\, \eta(t) = \frac{1}{2} \right\}$$

Notice $P(\tau < T) = 1$ since η is continuous and $\eta(T) = 0$. Now let $\{\Xi_i\}$ be any increasing sequence of stopping times dependent solely on Z_2 (independent of Z_1) with the properties

$$(\forall i = 1, \ldots, \infty)\, (\forall t \in [0,T])\; P(\Xi_i < t) > 0 \text{ and } P(\Xi_i < T) \downarrow 0 \text{ as } i \to \infty. \tag{5.36}$$

Let N be the random number defined by $\inf\{i : \Xi_i < \tau\} + 1$ if the infimum exists or by zero otherwise. The properties in (5.36) ensure N is well-defined, satisfies $N \geq 1$, and is finite almost surely but not uniformly bounded. The consumption dates along a given path ω in our example are $t_1(\omega) = \Xi_1(\omega) \wedge \tau(\omega)$,..., $t_i(\omega) = \Xi_i(\omega) \wedge \tau(\omega)$, ..., $t_{N(\omega)}(\omega) \equiv \tau(\omega)$, $t_{N(\omega)+1}(\omega) \equiv T$.

We use these consumption dates as the basis of a continuous-time overlapping generations model (Samuelson, 1958). A single representative investor represents each generation. The first, "generation 0," is endowed with one share of stock and participates in the financial market until the stopping time t_1, when it must consume from its financial wealth and depart from the economy. Any subsequent generation $i \geq 1$ arrives with the endowment

$$e_i = \frac{1}{2} + \frac{1}{2\eta(t_i)}.$$

and may trade until t_{i+1}, at which time it must consume from its financial wealth and depart from the economy. This process repeats with generation $i+1$ until $t_{i+1} = \tau$. The generation arriving at τ is the last, receives the endowment $e_\tau = \frac{1}{2} + \frac{1}{2\eta(\tau)} = \frac{3}{2}$, and consumes from its financial wealth at time T. The economy then ends. No generation may participate in the economy before the generation arrives or after it departs. The number of generations born along a given path ω is $N(\omega) < \infty$, and the total number of consumption dates is $N+1$. Our construction ensures $N+1$, the number of consumption dates and the number of generations, is finite almost surely but is not bounded by a constant.

Here is generation i's choice problem.

Choice Problem 5.1 (Generation i's Choice Problem). *On the time interval* $[t_i, t_{i+1}]$, *generation i chooses a portfolio* (π_B^i, π_S^i) *to maximize its expected utility*

$$E_{t_i}\left[\log\left(c^i(t_{i+1}) - \frac{1}{2}\right)\right],$$

subject to

$$W^i(t_i) = e_i,\; dW^i(t) = \pi_B^i(t)dB(t) + \pi_S^i(t)dS(t) + \pi_S^i(T)D(T)1_{\{t=T\}},\; W^i(t_{i+1}) \geq c^i(t_{i+1})$$

and $(\forall t \in [0,T])$ $W^i(t) \geq 0$, *P-almost surely. Expected utility equals minus infinity if* $P\big(c^i(t_{i+1}) \leq \frac{1}{2}\big) > 0$.

The preferences of each generation require it to have wealth in excess of 1/2 at its departure to avoid negative infinite utility. These preferences are meant to capture the spirit of "safety-first rules" (Roy, 1952).[11] Any generation born at time $t_i < \tau$ is uncertain whether it is the next to last generation, but generation τ knows it will be the last. In our example, no generation can hedge its departure time using the financial assets. The nonnegative wealth constraint serves to make doubling strategies infeasible; our results would be the same if negative wealth were bounded by any exogenous or endogenous negative number.

We prove below that an equilibrium price system for this example is $S(T) = 0$

$$S(t) = \frac{1}{2} + \frac{1}{2\eta(t \wedge \tau)} \quad \text{and} \quad B(t) \equiv 1. \tag{5.37}$$

Note $S(t_i) = e_i$ for all $i \leq N$ and $\lim_{t \to T} S(t) = D(T) = \frac{3}{2}$. This equilibrium stock price exceeds the lowest cost of replicating its terminal (and only) dividend of 3/2. Thus the stock has a bubble according to Definition 3.1, even though it is in positive net supply.[12] To see this, note that the initial stock price is 1. But with initial investment of 3/4, borrowing 3/4 at the locally riskless rate and buying 3/2 units of the stock also pays 3/2 at time T while maintaining nonnegative wealth of $3/(4\eta(t))$. No generation $i < N$ switches to the cheaper strategy because of the risk of needing to liquidate its portfolio prior to time τ (switching would yield a positive probability of wealth falling below $\frac{1}{2}$ and negative infinite expected utility).[13]

Proposition 4.1 implies Condition 4.1 is violated. Consistent with Proposition A.3, each generation i finances its consumption $c^i(t_{i+1}) = S(t_{i+1})$ at the lowest possible cost. Consistent with Theorem 4.1 there is no local asset pricing bubble on the stock. Our proof below shows there is a ν^* for which $\rho^{\nu^*}(t)S(t)$ is a martingale during the generation i's lifetime $[t_i, t_{i+1}]$, yet for all $\nu \in \mathcal{V}$ the process $\rho^\nu(t)S(t)$ is strictly a nonnegative local martingale (a supermartingale) over the intervals $[t_i, \tau]$ and $[t_i, T]$ when $i < N$. Thus a generation born at time $t_i < \tau$ that would happen to know for certain it would survive until τ would not optimally hold the stock, but no generation in our example has this knowledge.

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[11] The preferences also reflect aspects of goal-setting for intolerance for declines in standard of living (Dybvig, 1995), portfolio insurance (Leland, 1980; Grossman and Zhou, 1996), life-cycle concerns (Mariger, 1987), regulations requiring certain institutions to maintain liquid reserves, and mandated spending rules for university endowments (Dybvig, 1999).

[12] If we interpret $B(T)$ as a liquidating dividend, there is also a bond bubble, but this is less interesting because the bond is in zero net supply.

[13] Consistent with Proposition A.2, the ability to replicate the dividend at a lower cost implies $S(0) > \sup_{\nu \in \mathcal{V}} E[\rho^\nu(T)D(T)]$. To see this, first note that $\rho^0(t) = \eta(t \wedge \tau)$, and that each ρ^ν has the form

$$\rho^\nu(t) = \rho^0(t) \exp\left(-\frac{1}{2}\int_0^t \nu^2(s)ds - \int_0^t \nu(s)dZ_2(s)\right). \tag{5.38}$$

In particular, $\rho^0(T) = 1/2$, so $E[\rho^\nu(T)] \leq 1/2$. Moreover, $1 = S(0) > \frac{3}{4} \geq E[\rho^\nu(T)D(T)]$ for all $\nu \in \mathcal{V}$, which implies $S(0) > \sup_{\nu \in \mathcal{V}} E[\rho^\nu(T)D(T)]$.

The remainder of this section proves the equilibrium for the example.

Proof of the Equilibrium: Our candidate equilibrium prices are (5.37), and the candidate equilibrium strategy of each generation consists of buying and holding the stock and consuming its financial wealth. Clearly all markets clear given these strategies, so the remaining issue is whether the strategies maximize expected utility given the candidate equilibrium prices. This is clearly true for the generation which arrives at time τ so what remains is to show this for earlier generations.

We prove this for a given generation i that arrives at time $t_i < \tau$ and departs at time $t_{i+1} \leq \tau$. The proof has two steps. The first step shows that $S(t_i)$ is the lowest cost of obtaining the payout of $S(t_{i+1})$, which is necessary for the candidate strategy of buying and holding the stock to be optimal. The second step shows $S(t_{i+1})$ provides the highest expected utility for generation i given its budget constraint.

To perform the first step, we show there is a $\nu^* \in \mathcal{V}$ such that

$$E_{t_i}\left[\frac{\rho^{\nu^*}(t_{i+1})}{\rho^{\nu^*}(t_i)}S(t_{i+1})\right] = S(t_i),$$

for $t_i < \tau$ (see Proposition A.1). Recall that τ and Ξ_{i+1} are independent. In the equilibrium, $\rho^0(t) = \eta(t)$, which is independent of Ξ_{i+1}. Now define the process M by

$$M(t) = \frac{P_t(\Xi_{i+1} < T)}{P(\Xi_{i+1} < T)},$$

where P_t denotes the time-t conditional probability. This M has the following properties: it is a bounded martingale with $M(t) > 0$ for $t \in [0, T)$, its terminal value $M(T)$ is either zero or $1/P(\Xi_{i+1} < T)$, and $M(t)$ is independent of both $S(t)$ and $\rho^0(t)$ at any time $t \in [0, T]$. These properties imply the process $M\rho^0$ is a nonnegative local martingale and strictly positive prior to time T. It also follows that $M\rho^0 S$ is a nonnegative local martingale. By the Martingale Representation Theorem, there is a $\nu^* \in \mathcal{V}$ such that $\rho^{\nu^*}(t) = M(t)\rho^0(t)$ on the random interval $[0, \tau]$ (Protter, 1992, Theorem IV.3.42). Moreover, direct computation shows

$$\begin{aligned}
&E[\rho^{\nu^*}(\Xi_{i+1} \wedge \tau)S(\Xi_{i+1} \wedge \tau)] = E[M(\Xi_{i+1} \wedge \tau)\rho^0(\Xi_{i+1} \wedge \tau)S(\Xi_{i+1} \wedge \tau)] \\
&\quad = E[M(\Xi_{i+1} \wedge T)\rho^0(\Xi_{i+1} \wedge \tau)S(\Xi_{i+1} \wedge \tau)] = E[M(T)\rho^0(\Xi_{i+1} \wedge \tau)S(\Xi_{i+1} \wedge \tau)] \\
&\quad = \frac{1}{P(\Xi_{i+1} < T)}E[\rho^0(\Xi_{i+1} \wedge \tau)S(\Xi_{i+1} \wedge \tau)1_{\{\Xi_{i+1}<T\}}] \\
&\quad = \frac{1}{P(\Xi_{i+1} < T)}\left\{\frac{1}{2}P(\Xi_{i+1} < T) + \frac{1}{2}E[\eta(\Xi_{i+1} \wedge \tau)1_{\{\Xi_{i+1}<T\}}]\right\} \\
&\quad = \frac{1}{P(\Xi_{i+1} < T)}\left\{\frac{1}{2}P(\Xi_{i+1} < T) + \frac{1}{2}\int_0^T E[\eta(t \wedge \tau)]P(\Xi_{i+1} \in dt)dt\right\} = 1.
\end{aligned}$$

Since $\rho^{\nu^*}(0)S(0) = 1$, this shows $\rho^{\nu^*}S$ is a martingale on the interval $[0, \Xi_{i+1} \wedge \tau]$, so it is also a martingale on generation i's lifetime $[t_i, t_{i+1}]$ on $t_i < \tau$. By Proposition A.1, there is no feasible trading strategy that provides a higher payoff than $S(t_{i+1})$ at a lower cost than $S(t_i)$. We remark that both $\rho^{\nu^*}B$ and hence ρ^{ν^*} are both martingales on $[0, \Xi_{i+1} \wedge \tau]$ because $2\rho^{\nu^*}S \geq \rho^{\nu^*}B = \rho^{\nu^*} \geq 0$.

The second step shows $S(t_{i+1})$ maximizes generation i's expected utility given its endowment of $e_i = S(t_i)$. We will use the well-known inequality for concave functions: $u(x) - u(y) \geq u'(x)(x-y)$. In our case, $u(x) = \log(x - 1/2)$. For any trading strategy that satisfies $W(t_{i+1}) \geq 1/2$, we have $W(t) \geq 1/2$ for $t \in [t_i, t_{i+1}]$.[14] It follows that for any feasible trading strategy,

$$E_{t_i}\left[\rho^0(t_{i+1})\Big(W(t_{i+1}) - \frac{1}{2}\Big)\right] \leq \rho^0(t_i)\Big(e_i - \frac{1}{2}\Big).$$

Direct calculation shows equality holds when $W(t_{i+1}) = S(t_{i+1})$. Given the strategy of buying and holding the stock, generation i's marginal utility is $u'(S(t_{i+1})) = \rho^0(t_{i+1})/2$. Thus

$$\begin{aligned} E_{t_i}[u(S(t_{i+1}))] - E_{t_i}[u(W(t_{i+1}))] \\ \geq E_{t_i}\left[u'(S(t_{i+1}))\left(S(t_{i+1}) - \frac{1}{2} - \Big(W(t_{i+1}) - \frac{1}{2}\Big)\right)\right] \geq 0, \end{aligned}$$

so the equilibrium strategy does indeed maximize generation i's utility. □

5.2 Long Lived Investor

We now show how a failure of Condition 4.1 which allows bubbles can also allow long lived investor equilibrium wealth constraints to fail the transverality condition. We maintain all the previous assumptions of the model except we now assume the stock pays a liquidating dividend of $\frac{3}{2} + M$ at date T where $0 \leq M \leq \frac{1}{2}$. When M is changed, we are changing the aggregate endowment of the economy. If there is no other change in the model, then the equilibrium prices are the same except there will be a jump in the bond price and the stock price at time τ. The stock price jumps from $\frac{3}{2}$ to $\frac{3}{2} + M$ and to rule out arbitrage, the bond price must jump from 1 to $1 + \frac{2M}{3}$. The process $\rho^0(t)$ would also jump at time τ from $\frac{1}{2}$ to $\frac{3}{6+4M}$. These prices would then remain constant at these levels until time T. Although formally, we did not allow for this kind of jumps in our theoretical model, extensions to cover this are fairly minor. Since this is not our main focus, we will skip the formal derivation. Nevertheless, it is easy to see that $\rho^\nu(t)B(t)$ and $\rho^\nu(t)S(t)$ are still nonnegative local martingales.

We now introduce a long lived investor with no initial wealth and no endowments who solves the following choice problem. This long lived investor might be thought of as an institution like a hedge fund which does not require immediate liquidity until time T who can sustain mark to market losses as long as these losses do not exceed the exogenously specified amount M.

[14] This follows from the following observations: $\rho^{\nu^*}(t)W(t)$ is a nonnegative local martingale, thus a supermartingale. Therefore $W(t) \geq E_t[\frac{\rho^{\nu^*}(t_{i+1})}{\rho^{\nu^*}(t)}W(t_{i+1})] \geq 1/2$.

Choice Problem 5.2 (Long Lived Investor Choice Problem)**.** *Choose a portfolio* (π_B^L, π_S^L) *to maximize its expected utility*

$$E\left[u\left(c^L(T)\right)\right],$$

where $u : [0, \infty) \to \Re$ *is concave and strictly increasing, subject to*

$$W^L(0) = 0,\ dW^L(t) = \pi_B^L(t)dB(t) + \pi_S^L(t)dS(t) + \pi_S^L(T)D(T)1_{\{t=T\}},\ W^L(T) \geq c^L(T) \geq 0$$

and $(\forall t \in [0, T])$ $W^L(t) \geq a^L(t)$, *P-almost surely where for* $t \in [0, T)$ $a^L(t) = -MB(t)$, $a^L(T) = 0$. *This corresponds to the process* $g^L(t_i) = 0$ *for* $i < N + 1$, $g^L(T) = -MB(T)$.

The equilibrium discussed above is no longer an equilibrium when we add this long lived investor and $0 < M$. To see this, notice that the long lived investor can at time 0 buy M share of the stock and finance this purchase by selling M share of the bond. This portfolio has initial cost 0, and maintains wealth greater than or equal to $-MB(t)$, and provides a payoff of $M(\frac{3}{2} + M) - M(1 + \frac{2M}{3}) = \frac{M}{2} - \frac{M^2}{3}$. This payoff is strictly positive if $0 < M \leq \frac{1}{2}$. This would be inconsistent with market clearing.

There is an equilibrium when $0 \leq M \leq \frac{1}{2}$, however. Let $\pi = \frac{2M}{2M+1}$. Asset prices are given by $B(t) \equiv 1$, $S(t) = \frac{1}{2(1-\pi)} - \frac{\pi}{1-\pi} + \frac{1}{2(1-\pi)\eta(t)}$ for $t < T$ and $S(T) = 0$ (When $M = 0$ these are the same as the previous section). The process $\rho^0(t)$ is given by $\eta(t)$ as in the last section. Given these prices, it can be shown the long lived investor optimally chooses the portfolio $\pi_B^L(t) = -\pi$ and $\pi_S^L(t) = \pi$, this portfolio obeys the wealth constraint, and terminal consumption is given by $C^L(T) = M$.

The short lived investors optimally choose $\pi_B^i(t) = \pi$ and $\pi_S^i = 1 - \pi$. Given these choices, the wealth process for investor i satisfies $W^i(t_i) = e_i$ and $W^i(t_{i+1}) = e_{i+1}$. Our previous section analysis can be used to show this is optimal for each short lived investor(the process ρ^0, the endowments, and consumption choices are identical to those in the previous section).

Given these choices, markets obviously clear for $i < N + 1$. This last generation consumes $\frac{3}{2}$ at time T and the long lived investor consumes M at time T so markets clear at time T as well. In this equilibrium, the stock price satisfies

$$S(0) = 1 \geq \sup_\nu E[\rho^\nu(T)D(T)] = \frac{3}{4} + \frac{M}{2}$$

so the stock price has a long lived bubble when $M < \frac{1}{2}$ but when $M = \frac{1}{2}$ the stock price does not have a bubble. The bond price also has a bubble. Moreover, the long lived investor's wealth constraint does not satisfy a transversality constraint when $M > 0$ since

$$-a^L(0) = M > \sup_\nu E[-\rho^\nu(T)g^L(T)] = \sup_\nu E[\rho^\nu(T)M] = \frac{M}{2}. \tag{5.39}$$

This is inconsistent with the conclusions of Proposition 4.1. As in our previous section, Condition 4.1 is violated.

A Appendix: Proofs

A.1 Preliminary Results

In this section we collect several results which are used extensively in our analysis. The first result derives a local martingale property of wealth and bounds on negative wealth. The subsequent results concern the lowest cost of replicating a stream of cash flows given a lower bound on wealth.

Several of our proofs use the following lemma regarding the local martingale property of wealth and constraints on negative wealth.

Lemma A.1. *Let W be a wealth process which satisfies* (2.1). *Then*

$$\rho^\nu(t)W(t) + \int_{(0,t]} \rho^\nu(s)dC(s) - \sum_{j=1}^{N+1} \rho^\nu(t_j)e^i(t_j)1_{\{t_j \le t\}}$$

is a local martingale for every $\nu \in \mathcal{V}$. In addition, for any bound on negative wealth a^i satisfying Assumption 2.2, for given $A \in \mathcal{F}_{\varsigma_i}$, with $E[\rho^\nu(\varsigma_i)a^i(\varsigma_i)1_{\{A\}}] < \infty$ then

$$\rho^\nu(t)a^i(t)1_{\{A\}} + \sum_{j=1}^{N+1} \rho^\nu(t_j)g(t_j)1_{\{A \bigcap \{\varsigma_i < t_j \le t\}\}}$$

is a local martingale for $t \ge \varsigma_i$.

Proof. Let $\nu \in \mathcal{V}$ be given. Ito's Lemma implies

$$\rho^\nu(t)W(t) + \int_{(0,t]} \rho^\nu(s)dC(s) - \sum_{j=1}^{N+1} \rho^\nu(t_j)e^i(t_j)1_{\{t_j \le t\}}$$
$$= W(0) + \int_0^t \rho^\nu(s)\Big(\tilde{\pi}_S(s)\sigma(s) - W(s)\big(\theta'(s) + \nu(s)\big)\Big)dZ(s), \quad \text{(A.1)}$$

where the K-dimensional row vector $\tilde{\pi}_S$ represents the dollar investments in the K risky assets (i.e., $\tilde{\pi}(t) = (\pi_{Sk}(t)S_k(t))_{k=1,...K}$). (For a similar calculation, see, for example, Loewenstein and Willard (2000b).) The integral in (A.1) is locally bounded, so it is a local martingale (Karatzas and Shreve, 1988, Chapter 3). Assumption 2.2 and Ito's lemma also imply

$$\rho^\nu(t)a^i(t) + \sum_{j=1}^{N+1} \rho^\nu(t_j)g(t_j)1_{\{\varsigma_i < t_j \le t\}} =$$
$$\rho^\nu(\varsigma_i)a^i(\varsigma_i) + \int_{\varsigma_i}^t \rho^\nu(s)\Big(\tilde{\alpha}_S(s)\sigma(s) - a^i(s)\big(\theta'(s) + \nu(s)\big)\Big)dZ(s), \quad \text{(A.2)}$$

where the K-dimensional row vector $\tilde{\alpha}_S$ represents the dollar investments in the K risky assets (i.e., $\tilde{\alpha}_S(t) = (\alpha_k(t)S_k(t))_{k=1,...K}$). The integral in (A.2) is locally bounded, so it is a local martingale (Karatzas and Shreve, 1988, Chapter 3). Thus if for given $A \in \mathcal{F}_{\varsigma_i}$, $E[\rho^\nu(\varsigma_i)a^i(\varsigma_i)1_{\{A\}}] < \infty$ then $\rho^\nu(t)a^i(t)1_{\{A\}} + \sum_{j=1}^{N+1} \rho^\nu(t_j)g(t_j)1_{\{A \bigcap \{\varsigma_i < t_j \le t\}\}}$ is a local martingale for $t \ge \varsigma_i$. $\square$

Proposition A.1. *Given Assumption 3.2 and a nonpositive process a describing a bound on negative wealth that satisfies Assumption 2.2 with $\varsigma_i = 0$, and a nonnegative progressively measurable process γ with $\gamma(t_j) \geq g(t_j)$, a $\mathcal{F}_{t_n \wedge T}$ measurable random variable X which satisfies $X \geq a(t_n \wedge T)$, and*

$$\sup_{\nu \in \mathcal{V}} E\Big[\sum_{j=1}^{n\wedge N+1} \rho^\nu(t_j)\big(\gamma(t_j) - g(t_j)\big) + \rho^\nu(t_n \wedge T)\big(X - a(t_n \wedge T)\big)\Big] = w - a(0), \tag{A.3}$$

then there is a trading strategy with wealth W and $W(0) = w$, $W(t_n \wedge T) \geq X$, P-almost surely, and $W(t) \geq a(t)$ pathwise on $[0, t_n \wedge T]$ which satisfies (2.1) with $C(t) \geq \sum_{0<t_j\leq t} \gamma(t_j)$ for $0 \leq t \leq t_n \wedge T$ and $e^i = 0$. Moreover, for any stopping time $\chi \leq t_n \wedge T$, let $W(\chi)$ be defined by

$$W(\chi) =$$

$$a(\chi) + \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_t\Big[\sum_{j=1}^{n\wedge N+1} \rho^\nu(t_j)\big(\gamma(t_j) - g(t_j)1_{\{t_j>\chi\}}\big) + \rho^\nu(t_n\wedge T)\big(X - a(t_n \wedge T)\big)\Big]}{\rho^\nu(\chi)} \tag{A.4}$$

where essup *denotes essential supremum.*[15] *Then then there is a trading strategy with wealth at time χ equal to $W(\chi)$, $W(t_n \wedge T) \geq X$, P-almost surely, and $W(t) \geq a(t)$ pathwise on $[\chi, t_n \wedge T]$ which satisfies (2.2) on $[\chi, t_n \wedge T]$ with $C(t) - C(\chi) \geq \sum_{\chi<t_j\leq t} \gamma(t_j)$ for $0 \leq t \leq t_n \wedge T$ and $e^i = 0$.*

Proof of Proposition A.1. For $t \leq t_n \wedge T$ define

$$\hat{W}(t) = \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_t\big[\sum_{j=1}^{n\wedge N+1} \rho^\nu(t_j)\left(\gamma(t_j) - g(t_j)\right)1_{\{t_j>t\}} + \rho^\nu(t_n\wedge T)\left(X - a(t_n\wedge T)\right)\big]}{\rho^\nu(t)}. \tag{A.5}$$

and let $\tau = \inf\{t \leq t_n \wedge T | \hat{W}(t) = 0\}$. For any $\nu \in \mathcal{V}$, there is a modification of $\rho^\nu(t\wedge\tau)\hat{W}(t\wedge\tau) + \sum_{\{t_j\leq t\wedge\tau\}} \rho^\nu(t_j)\left(\gamma(t_j) - g(t_j)\right)$ that is a RCLL supermartingale, which we continue to denote by $\rho^\nu(t)\hat{W}(t)$ (the proof is virtually identical to Karatzas and Shreve (1998, Theorem 5.6.5)[16]). The Doob-Meyer decomposition and the Martingale Representation Theorem imply

$$\rho^\nu(t\wedge\tau)\hat{W}(t\wedge\tau) + \sum_{0<t_j\leq t\wedge\tau} \rho^\nu(t_j)\left(c(t_j) - g(t_j)\right) = \hat{W}(0) + \int_0^{t\wedge\tau} \psi^\nu(s)dZ(s) - A^\nu(t\wedge\tau), \tag{A.6}$$

[15] Essential supremum describes the least upper bound for a set of random variables. The essential supremum of a family of measurable functions $\{g_\lambda, \lambda \in \Lambda\}$ is denoted by $g = \operatorname{essup}_{\lambda\in\Lambda} g_\lambda$ and is defined by (i) g is measurable, (ii) $g \geq g_\lambda$ for all $\lambda \in \Lambda$, and (iii) for any h satisfying (i) and (ii), $h \geq g$ (Chow and Teicher (1997)). In our setting, a given set over which we take essential supremum will be directed upwards (so the essential supremum over $\mathcal{V}$ can be approximated by an increasing sequence of elements from the set under consideration).

[16] Observe for a fixed $\rho^{\hat{\nu}}$ we have

$$\begin{aligned}\hat{W}(t\wedge\tau) &= \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_{t\wedge\tau}\big[\sum_{j=1}^{n\wedge N+1} \rho^\nu(t_j)\left(\gamma(t_j) - g(t_j)\right)1_{\{t_j>t\}} + \rho^\nu(t_n\wedge T)\left(X - a(t_n\wedge T)\right)\big]}{\rho^\nu(t\wedge\tau)} \\ &\geq \frac{E_{t\wedge\tau}\big[\sum_{j=1}^{n\wedge N+1} \rho^{\hat{\nu}}(t_j)\left(\gamma(t_j) - g(t_j)\right)1_{\{s\geq t_j>t\}} + \rho^{\hat{\nu}}(s\wedge\tau)\hat{W}(s\wedge T)\big]}{\rho^{\hat{\nu}}(t\wedge\tau)}.\end{aligned}$$

where ψ^ν is progressively measurable and A^ν is a nondecreasing finite-variation process. Calculations virtually identical to those by Karatzas and Shreve (1998, pages 217-218) show both

$$\phi(t) \equiv \frac{\psi^\nu(t)}{\rho^\nu(t)} + \hat{W}(t)(\theta'(t) + \nu(t)) \quad \text{on} \quad t \leq \tau \tag{A.7}$$

and

$$\hat{C}(t \wedge \tau) \equiv \int_{(0,t\wedge\tau]} \frac{dA^\nu(s)}{\rho^\nu(s)} - \int_0^{t\wedge\tau} \phi(s)\nu'(s)ds \tag{A.8}$$

are independent of ν. Taking $\nu \equiv 0$ shows C is non-decreasing. We now show there is a progressively measurable trading strategy $\tilde{\pi}_S$ satisfying $\tilde{\pi}_S(t)\sigma(t) = \phi(t)$, using arguments similar to those of Karatzas and Shreve (1998, page 219). We first note (A.8) implies

$$\int_{(0,t\wedge\tau]} \frac{dA^\nu(s)}{\rho^\nu(s)} = \hat{C}(t \wedge \tau) + \int_0^{t\wedge\tau} \phi(s)\nu'(s)ds \geq 0 \tag{A.9}$$

for all $t \in [0, \tau]$ because A^ν is nondecreasing. We show $\phi'(t)$ is in the range of $\sigma'(t)$ for all $t \in [0, \tau]$. To this end, we first choose $\hat{\nu}(t)$ to be the process defined by the orthogonal projection of $\phi'(t)$ onto the null space of $\sigma(t)$ at each time $t \in [0, \tau]$. The process $\hat{\nu}$ is progressively measurable (Karatzas and Shreve, 1998, Corollary 1.4.5), and it satisfies $\sigma\hat{\nu}' \equiv 0$. Thus $\gamma\hat{\nu} \in \mathcal{V}$ for all real valued numbers γ. Substituting $\hat{\nu}$ into the second integral in (A.9), we get $\gamma \int_0^{t\wedge\tau} \|\hat{\nu}(s)\|^2 ds$. Assuming $\tau > 0$ (otherwise there is nothing to prove), this integral would be nonzero if both $\hat{\nu}$ and γ are. Choosing γ to be sufficiently negative would make the left-hand quantity in (A.9) negative, a contradiction. Thus $\hat{\nu}$ must in fact be identically zero, which implies $\phi'(t)$ is in the range of $\sigma'(t)$ for Lebesgue $\times$ P all $t \in [0, \tau]$ (equivalently, $\phi(t)$ is in the orthogonal complement of the nullspace of $\sigma(t)$). The existence of a progressively measurable $\tilde{\pi}_S$ satisfying $\tilde{\pi}_S(t)\sigma(t) = \phi(t)$ then follows from Karatzas and Shreve (1998, Lemma 1.4.7).

Setting $\nu = 0$ in (A.6) and (A.7), we have

$$\begin{aligned} \rho^0(t \wedge \tau)\hat{W}(t \wedge \tau) &+ \sum_{\{t_j \leq t\}} \rho^0(t_j)\left(\gamma(t_j) - g(t_j)\right) \\ &= W(0) + \int_0^{t\wedge\tau} \rho^0(s)[\sigma'(s)\tilde{\pi}_S'(s) - \hat{W}(s)\theta(s)]'dZ(s) - \int_{(0,t\wedge\tau]} \rho^0(s)d\hat{C}(s). \end{aligned} \tag{A.10}$$

Comparing (A.10) with (A.1), we see $\hat{W}$ is a nonnegative wealth process that starts with $\hat{W}(0) = w - a(0)$, the dollar investment in the risky assets is given by the $1 \times K$ row vector $\tilde{\pi}_S$, invests $\hat{W} - \tilde{\pi}_S \mathbf{1}_K$ in the bond, with cumulative consumption $C(t) = \hat{C}(t) + \sum_{0<t_j\leq t} \gamma(t_j) - g(t_j)$ for $t \leq t_n \wedge T$, $e^i \equiv 0$ and has terminal payoff $\hat{W}(\tau) = X - a(t_n \wedge T)$.

Let $W = \hat{W} + a$. Then $W \geq a$ and corresponds to a wealth process which invests $\tilde{\pi}_S + \tilde{\alpha}_S$ in the risky assets (where $\tilde{\alpha}_S$ represents the $1 \times K$ row vector of dollar amounts invested in

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which implies

$$\begin{aligned} \rho^{\hat{\nu}}(t \wedge \tau)\hat{W}(t \wedge \tau) &+ \sum_{0<t_j\leq t\wedge\tau} \rho^{\hat{\nu}}(t_j)\left(\gamma(t_j) - g(t_j)\right) \\ &\geq E_{t\wedge\tau}\Big[\sum_{0<t_j\leq s\wedge\tau} \rho^{\hat{\nu}}(t_j)\left(\gamma(t_j) - g(t_j)\right) + \rho^{\hat{\nu}}(s \wedge \tau)\hat{W}(s \wedge \tau)\Big] \end{aligned}$$

risky assets in the portfolio describing the wealth constraint), $\hat{W}+a-\tilde{\pi}_S\mathbf{1}_K-\tilde{\alpha}_S\mathbf{1}_K$ in the bond, with cumulative consumption $C(t)=\hat{C}(t)+\sum_{0<t_j\le t}\gamma(t_j)$ for $t\le t_n\wedge T$, with $e^i\equiv 0$ and has terminal payoff $W(\tau)=X$. By construction this portfolio requires initial wealth

$$w=a(0)+\sup_{\nu\in\mathcal{V}}E\big[\sum_{j=1}^{n\wedge N+1}\rho^\nu(t_j)\,(\gamma(t_j)-g(t_j))+\rho^\nu(t_n\wedge T)\,(X-a(t_n\wedge T))\big].$$

and by construction

$$W(\chi)=a(\chi)+\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_t\left[\sum_{j=1}^{n\wedge N+1}\rho^\nu(t_j)\big(\gamma(t_j)-g(t_j)1_{\{t_j>\chi\}}\big)+\rho^\nu(t_n\wedge T)\big(X-a(t_n\wedge T)\big)\right]}{\rho^\nu(\chi)}$$

□

Proposition A.2. *Given a wealth constraint $a^i(t)$ which satisfies Assumption 2.2 with $\varsigma_i=0$, consider a nonnegative progressively measurable process γ with*

$$\sup_{\nu\in\mathcal{V}}E\left[\sum_{j=1}^{N+1}\rho^\nu(t_j)\gamma(t_j)\right]<\infty. \tag{A.11}$$

Given Assumption 3.2, and a stopping time $\chi<T$ define

$$W(\chi)=a^i(\chi)+\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\chi\left[\sum_{j=1}^{N+1}\rho^\nu(t_j)\big(\gamma(t_j)-g^i(t_j)\big)1_{\{t_j>\chi\}}\right]}{\rho^\nu(\chi)} \tag{A.12}$$

Then then there is a trading strategy with wealth at time χ equal to $W(\chi)$, $W(t)\ge a(t)$ pathwise on $[\chi,t_n\wedge T]$ for all n, which satisfies (2.2) on $[\chi,t_n\wedge T]$ for all n, with $C(t)-C(\chi)\ge\sum_{\chi<t_j\le t}\gamma(t_j)$ for $0\le t\le t_n\wedge T$ and all n, and $e^i=0$.

Proof of Proposition A.2. For each n let

$$X_n=\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_{t_n}\left[\sum_{j=1}^{N+1}\rho^\nu(t_j)\,(\gamma(t_j)-g(t_j))\,1_{\{t_j>t_n\}}\right]}{\rho^\nu(t_n)}1_{\{t_n\le T\}}$$

and let

$$W^n(t)=\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_t\left[\sum_{j=1}^{n\wedge N+1}\rho^\nu(t_j)\,(\gamma(t_j)-g(t_j))\,1_{\{t_j>t\}}+\rho^\nu(t_n\wedge T)X_n\right]}{\rho^\nu(t)}$$

Then as in the proof of Proposition A.1 Statement 2 (set $a\equiv g\equiv 0$ for the a and g that appear there and let the γ which appears there to be $\gamma-g$.) W^n is a wealth process with a corresponding portfolio $\pi^n=[\pi_B^n,\pi_S^n]$ which satisfies (2.1) with $e^i\equiv 0$, with cumulative consumption $C(t)=\hat{C}(t)+\sum_{0<t_j\le t}\gamma(t_j)-g(t_j)$ for $t\le t_n\wedge T$, $W^n(t_n\wedge T)=X_n$ and maintains $W^n(t)\ge 0$.

Arguments similar to those in Karatzas and Shreve (1998, Theorem 5.6.5) give the equation of dynamic programming for $t\le t_n\wedge T$

$$\begin{aligned}W^n(t)&=\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_t\left[\sum_{j=1}^{n\wedge N+1}\rho^\nu(t_j)\,(\gamma(t_j)-g(t_j))\,1_{\{t_j>t\}}+\rho^\nu(t_n\wedge T)X_n\right]}{\rho^\nu(t)}\\&=\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_t\left[\sum_{j=1}^{N+1}\rho^\nu(t_j)\,(\gamma(t_j)-g(t_j))\,1_{\{t_j>t\}}\right]}{\rho^\nu(t)}\end{aligned}$$

Now let $\pi_B(t) = \sum_{n=1}^{\infty} \pi_B^n(t) 1_{\{t_{n-1} \le t < t_n \wedge T\}}$ and $\pi_S(t) = \sum_{n=1}^{\infty} \pi_S^n(t) 1_{\{t_{n-1} \le t < t_n \wedge T\}}$. This strategy then satisfies (2.1) with $e^i \equiv 0$, with cumulative consumption $C(t) = \hat{C}(t) + \sum_{0<t_j \le t} \gamma(t_j) - g(t_j)$ for $t \le t_n \wedge T$ for all n, with $W(t) \ge 0$ on $[0, t_n \wedge T]$ for all n. Moreover, letting $\tilde{W}(t) = \pi_B(t)B(t) + \pi_S(t)S(t)$

$$\tilde{W}(t) = \mathrm{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\left(\gamma(t_j) - g(t_j)\right) 1_{\{t_j > t\}}\right]}{\rho^\nu(t)}$$

Now observe the strategy $W(t) = \tilde{W}(t) + a(t)$ satisfies (2.1) with $e^i \equiv 0$, with cumulative consumption $C(t) = \hat{C}(t) + \sum_{0<t_j \le t} \gamma(t_j)$ for $t \le t_n \wedge T$ for all n, with $W(t) \ge a(t)$ on $[0, t_n \wedge T]$ for all n. We have

$$W(t) = a(t) + \mathrm{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\left(\gamma(t_j) - g(t_j)\right) 1_{\{t_j > t\}}\right]}{\rho^\nu(t)}$$

□

A.2 Proofs for Section 3.1

Proof of Proposition 3.1. Given $\pi_B^i(0) = 0$, $\pi_S^i(0) = 0$, $e^i = 0$, the budget equation at time 0 implies $W(0) = -C(0)$. The nonegative wealth constraint implies $W(0) \ge 0$, so it follows $W(0) = C(0) = 0$. Lemma A.1 with $\nu \equiv 0$ and $e^i \equiv 0$ implies for such a strategy the process $\rho^0(t)W(t) + \int_{(0,t]} \rho^0(s)dC(s)$ is a local martingale. It is also nonnegative so it is a supermartingale. Therefore

$$E\left[\int_{(0,t_n \wedge T]} \rho^0(s)dC(s)\right] \le E\left[\int_{(0,t_n \wedge T]} \rho^0(s)dC(s) + \rho^0(t_n \wedge T)W(t_n \wedge T)\right] \le W(0) = 0 \tag{A.13}$$

Since $\rho^0(t) > 0$ and C is nondecreasing this implies $C(t) = 0$ for $t \in [0, t_n \wedge T]$ for all n almost surely. □

Proof of Proposition 3.2. The proof follows directly from Proposition A.1: take $a \equiv 0$, $X \equiv S_k^{\mathrm{ex}}(t_n \wedge T)$, $\gamma(t_j) = D_k(t_j)$, and $w = \sup_{\nu \in \mathcal{V}} E\left[\sum_{j=1}^{n \wedge N+1} \rho^\nu(t_j)D_k(t_j) + \rho^\nu(t_n \wedge T)S_k^{\mathrm{ex}}(t_n \wedge T)\right]$. □

Proof of Proposition 3.6. Given that a satisfies Assumption 2.2, the process $-\rho^\nu(t)a^i(t) - \sum_{j=1}^{N+1} \rho^\nu(t_j)g^i(t_j)1_{\{\varsigma_i < t_j \le t\}}$ is a nonnegative local martingale on $\varsigma_i \le t$ and hence a supermartingale for all $\nu \in \mathcal{V}$ from Lemma A.1.

Given inequality (3.20), consider the payout at $t_n \wedge \tau_i$

$$X^a = \frac{\rho^0(\varsigma_i)\left(-a^i(\varsigma_i)1_{\{A\}} - \mathrm{essup}_{\nu \in \mathcal{V}} E_{\varsigma_i}\left[-\frac{\rho^\nu(t_n \wedge \tau_i)}{\rho^\nu(\varsigma_i)} a^i(t_n \wedge \tau_i)1_{\{A\}}\right]\right)}{\rho^0(t_n \wedge \tau_i)} > 0$$

Because

$$E_{\varsigma_i}\left[\frac{\rho^0(\varsigma_i)\rho^\nu(t_n \wedge \tau_i)}{\rho^\nu(\varsigma_i)\rho^0(t_n \wedge \tau_i)}\right] \le 1$$

for all $\nu \in \mathcal{V}$, we have

$$\sup_{\nu \in \mathcal{V}} E\left[\rho^\nu(t_n \wedge \tau_i)\left(X - a^i(t_n \wedge \tau_i)\right) 1_{\{A\}}\right] \le \sup_{\nu \in \mathcal{V}} E\left[-\rho^\nu(\varsigma_i)a^i(\varsigma_i)1_{\{A\}}\right] \tag{A.14}$$

Let $\gamma(t_j) = -g^i(t_j)$ and $X = X^a - a^i(t_n \wedge \tau_i)1_{\{A\}}$ in Proposition A.1 and set the a and g in the proposition to be identically 0. This then implies implies there is a trading strategy that requires initial wealth $W(0) \le \sup_{\nu\in\mathcal{V}} E\big[-\rho^\nu(\varsigma_i)a^i(\varsigma_i)1_{\{A\}}\big]$, satisfies 2.1 with $e^i = 0$, with cumulative consumption $C(t) \ge \sum_{j=1}^{n\wedge N+1} -g^i(t_j)1_{\{t_j\le t\}}$, and terminal payout $\tilde{W}(t_n \wedge \tau) \ge X > 0$, and satisfies $\tilde{W}(t) \ge 0$. This wealth process has value

$$\tilde{W}(t) = \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\sum_{j=1}^{n\wedge N+1} -\rho^\nu(t_j)g^i(t_j)1_{\{t_j>t\}} + \rho^\nu(t_n\wedge T)X^a\right]}{\rho^\nu(t)}. \tag{A.15}$$

For $t \in [\varsigma_i, t_n \wedge T]$ define $W(t) = \tilde{W}(t) + a^i(t)$. This wealth process then requires no initial wealth, satisfies (2.1) with $e^i = 0$ and $C(t) \ge 0$, maintains $W(t) \ge a^i(t)$ and has terminal payoff $W(t_n \wedge T) = X^a > 0$. □

Proof of Proposition 3.5. Let $0 < b < 1$ and $\gamma(t_j) = b^j \frac{1}{\rho^0(t_j)} > 0$ for $t_j \le T$. Observe

$$\sup_{\nu\in\mathcal{V}} E\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\gamma(t_j)\right] \le \frac{1}{1-b} < \infty$$

Then Proposition A.2 indicates if we define

$$W(0) = a^i(0) + \sup_{\nu\in\mathcal{V}} E\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\left(\gamma(t_j) - g^i(t_j)\right)\right]$$

Then there exists a trading strategy, a cumulative consumption process $C(t) \ge \sum_{j=1}^{N+1} \gamma(t_j)1_{\{t_j\le t\}}$, such that the wealth process satifies 2.1 and maintains $W(t) \ge a^i(t)$ for $t \in [0, t_n \wedge T]$ and all n. Observe the initial wealth required for this strategy satisfies

$$W(0) = a^i(0) + \sup_{\nu\in\mathcal{V}} E\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\left(\gamma(t_j) - g^i(t_j)\right)\right] \le \frac{1}{1-b} + a^i(0) + \sup_{\nu\in\mathcal{V}} E\left[\sum_{j=1}^{N+1} -\rho^\nu(t_j)g^i(t_j)\right] \tag{A.16}$$

Given the inequality in (3.18) choose b so the right hand side is is zero. Then this strategy satisfies the conditions in the Proposition. □

A.3 Proofs for Section 4

A.3.1 Proof of Theorem 4.1

We now describe the economic steps of our proof of Theorem 4.1 (the more mathematical details appear in Section A.3.3). Our assumption that all investors prefer more to less implies they invest no more than necessary to finance consumption, as we now show.

Proposition A.3. *Assume an equilibrium exists with Assumptions 2.1 and 3.2. Given either an exogenous constraint a^i on negative wealth in Problem 2.1 or the equilibrium endogenous constraint a^i in Problem 2.2, the wealth of each investor $i \in \mathcal{I}_{n-1}$ on $t_{n-1} < T$ satisfies for each $A \in \mathcal{F}_{t_{n-1}} \cap \{i \in \mathcal{I}_{n-1}\}$ with $\sup_{\nu\in\mathcal{V}} E[\rho^\nu(t_{n-1})(W^i(t_{n-1}) - a^i(t_{n-1}))1_{\{A\}}] < \infty$*

$$(W^i(t) - a^i(t))1_{\{A\}} = \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)\big(W^{i,cum}(t_n) - a^{i,cum}(t_n)\big)1_{\{A\}}\right]}{\rho^\nu(t)}, \tag{A.17}$$

where

$$W^{i,cum}(t_n) = W^i(t_n) + c^i(t_n) - e^i(t_n),$$
$$a^{i,cum}(t_n) = a^i(t_n) + g^i(t_n),$$

and essup *denotes essential supremum.*

Proof. See Appendix A.3.3. □

Remark A.1. *Since our assumptions only allow us to assume W^i and a^i are finite almost surely we have to introduce the subset A to make sure the conditional expectations are well defined. This only causes minor difficulty however. There exists A_M with $\bigcup_M A_M = \Omega$ such that the statements in the Proposition hold on each A_M. For example, for $M > 0$, let $A_M = \{\omega | \rho^0(t_{n-1})(W^i(t_{n-1}) - a^i(t_{n-1})) \geq M\} \bigcap \{i \in \mathcal{I}_{n-1}\}$.*

Given that monotone investors finance their consumption at the lowest cost, we now use market clearing and the constraints on negative wealth to establish an *upper* bound on each investor's wealth. These upper bounds prevent aggregate financial wealth from growing large enough to support bubbles on assets that contribute to it (those in positive net supply).

Let $A_M \in \mathcal{F}_{t_{n-1}} \bigcap \{t_{n-1} < T\}$ with $E[\rho^\nu(t_{n-1}) \sum_{i\in\mathcal{I}_{n-1}} W^i(t_{n-1}) - a^i(t_{n-1}) 1_{\{A_M\}}] < \infty$ and $\bigcup_M A_M = \Omega$. Because every investor's equilibrium financial wealth satisfies (A.17), clearing the consumption market bounds the wealth for every investor $i \in \mathcal{I}_{n-1}$ by the largest possible value of the aggregate dividends $\bar{\pi}_S D(t_n)$ plus the aggregate financial wealth $\bar{\pi}_S S^{\text{ex}}(t_n)$ plus the absolute value of the aggregate allowable terminal negative wealth $-\sum_{i\in\mathcal{I}_{n-1}} a^i(t_n) + g^i(t_n)$. This follows from the following inequalities:

$$\begin{aligned}
W^i(t) 1_{\{A_M \bigcap \{i\in\mathcal{I}_{n-1}\}\}} &\leq (W^i(t) - a^i(t)) 1_{\{A_M \bigcap \{i\in\mathcal{I}_{n-1}\}\}} \\
&= \text{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)\big(W^{i,\text{cum}}(t_n) - a^{i,\text{cum}}(t_n)\big) 1_{\{A_M \bigcap \{i\in\mathcal{I}_{n-1}\}\}}\right]}{\rho^\nu(t)} \\
&\leq \text{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\rho^\nu(t_n) \sum_{i\in\mathcal{I}_{n-1}} \big(W^{i,\text{cum}}(t_n) - a^{i,\text{cum}}(t_n)\big) 1_{\{A_M \bigcap \{i\in\mathcal{I}_{n-1}\}\}}\right]}{\rho^\nu(t)} \\
&= \text{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)\big(\bar{\pi}_S S^{\text{ex}}(t_n) + \bar{\pi}_S D(t_n) - \sum_{i\in\mathcal{I}_{n-1}} (a^i(t_n) + g^i(t_n))\big) 1_{\{A_M \bigcap \{i\in\mathcal{I}_{n-1}\}\}}\right]}{\rho^\nu(t)}.
\end{aligned} \tag{A.18}$$

Now we show that clearing the asset market bounds aggregate financial wealth in a manner that rules out bubbles on positive net supply assets. This bound relies on our assumption that the number of investors participating in the market at any time is no more than I. Specifically, we have

$$\begin{aligned}
0 &\leq \sum_{i\in\mathcal{I}_{n-1}} \big(W^i(t) - a^i(t)\big) 1_{\{A_M\}} = (\bar{\pi}_S S^{\text{ex}}(t) - \sum_{i\in\mathcal{I}_{n-1}} a^i(t)) 1_{\{A_M\}} \\
&\leq I \times \text{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)\big(\bar{\pi}_S S^{\text{ex}}(t_n) + \bar{\pi}_S D(t_n) - \sum_{i\in\mathcal{I}_{n-1}} (a^i(t_n) + g^i(t_n))\big) 1_{\{A_M\}}\right]}{\rho^\nu(t)}.
\end{aligned} \tag{A.19}$$

We state a useful mathematical result.

Lemma A.2. *Let $A \in \mathcal{F}_{t_{n-1}} \bigcap \{t_{n-1} < T\}$ and X be a nonnegative process such that $\rho^\nu X 1_{\{A\}}$ is a local martingale on $[t_{n-1}, t_n]$ for all $\nu \in \mathcal{V}$. Suppose*

$$X(t) \leq \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)\Lambda 1_{\{A\}}\right]}{\rho^\nu(t)}$$

for some nonnegative $\mathcal{F}_{t_n}$ measurable random variable Λ with $\sup_{\nu \in \mathcal{V}} E[\rho^\nu(t_n)\Lambda 1_{\{A\}}] < \infty$. Then for $t \in [t_{n-1}, t_n]$

$$X(t)1_{\{A\}} = \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(t)},$$

and there exists a $\nu^* \in \mathcal{V}$ such that $\rho^{\nu^*}(t)X(t)1_{\{A\}}$ is a martingale on $[t_{n-1}, t_n]$.

Proof. See Appendix A.3.3. □

Taking

$$\Lambda = I \times (\bar{\pi}_S(D(t_n) + S^{\text{ex}}(t_n)) - \sum_{i \in \mathcal{I}_{n-1}} (a^i(t_n) + g^i(t_n)))1_{\{A_M\}}$$

in Lemma A.2. Let $X(t) = \left(\bar{\pi}_S S^{\text{ex}}(t) + \bar{\pi}_S D(t_n)1_{\{t=t_n\}} - \sum_{i \in \mathcal{I}_{n-1}} \left(a^i(t) + g^i(t_n)1_{\{t=t_n\}}\right)\right)1_{\{A_M\}}$. Lemma A.2 then says on $t_{n-1} \leq t < t_n$

$$\begin{aligned} &\left(\bar{\pi}_S S^{\text{ex}}(t) - \sum_{i \in \mathcal{I}_{n-1}} a^i(t)\right)1_{\{A_M\}} \\ &= \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)(\bar{\pi}_S(D(t_n) + S^{\text{ex}}(t_n)) - \sum_{i \in \mathcal{I}_{n-1}}(a^i(t_n) + g^i(t_n)))1_{\{A_M\}}\right]}{\rho^\nu(t)}. \end{aligned} \tag{A.20}$$

Since

$$S_k^{\text{ex}}(t)1_{\{A_M\}} \geq \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)((D_k(t_n) + S_k^{\text{ex}}(t_n))1_{\{A_M\}}\right]}{\rho^\nu(t)}$$

$$-a^i(t)1_{\{A_M\}} \geq \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[-\rho^\nu(t_n)(g^i(t_n) + a^i(t_n))1_{\{A_M\}}\right]}{\rho^\nu(t)}$$

then (A.20) holds for $t \in [t_{n-1}, t_n)$ only if

$$S_k^{\text{ex}}(t)1_{\{A_M\}} = \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)((D_k(t_n) + S_k^{\text{ex}}(t_n))1_{\{A_M\}}\right]}{\rho^\nu(t)}$$

whenever $\bar{\pi}_k > 0$ and

$$-a^i(t)1_{\{A_M\}} = \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[-\rho^\nu(t_n)(g^i(t_n) + a^i(t_n))1_{\{A_M\}}\right]}{\rho^\nu(t)}$$

for every equilibrium a^i and $i \in \mathcal{I}_{n-1}$. Lemma A.2 also implies there exists a $\nu^* \in \mathcal{V}$ such that for $t \in [t_{n-1}, t_n)$

$$S_k^{\text{ex}}(t)1_{\{A_M\}} = \frac{E_t\left[\rho^{\nu^*}(t_n)((D_k(t_n) + S_k^{\text{ex}}(t_n))1_{\{A_M\}}\right]}{\rho^\nu(t)}$$

whenever $\bar{\pi}_k > 0$ and

$$a^i(t)1_{\{A_M\}} = \frac{E_t\left[\rho^{\nu^*}(t_n)(g^i(t_n) + a^i(t_n))1_{\{A_M\}}\right]}{\rho^{\nu^*}(t)}$$

for every equilibrium a^i and $i \in \mathcal{I}_{n-1}$.

Theorem 4.1 then follows from these observations and the fact $\bigcup_M A_M = \Omega$.

A.3.2 Proof of Proposition 4.1

We will assume $P\{N \geq n\} > 0$ for all n since otherwise Theorem 4.1 implies the result. We show in Proposition A.2 that Condition 4.1 implies there is a finite-cost portfolio with payouts that superreplicate the payout stream $\{\gamma_1, \ldots\}$. This portfolio maintains nonnegative wealth.

First observe Theorem 4.1 implies (since each short lived investor participates over a uniformly bounded number of consumption dates) each short lived investor's wealth constraint must satisfy

$$-a^i(\varsigma_i)1_{\{A\}} = \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_{\varsigma_i}\left[\left(\sum_{j=1}^{N+1} \rho^\nu(t_j) g^i(t_j) 1_{\{\varsigma_i < t_j\}}\right) 1_{\{A\}}\right]}{\rho^\nu(\varsigma_i)} \tag{A.21}$$

on each $A \in \mathcal{F}_{\varsigma_i}$ with $\sup_{\nu\in\mathcal{V}} E[-\rho^\nu(\varsigma_i) a^i(\varsigma_i) 1_{\{A\}}] < \infty$. For these investors, financial wealth must satisfy

$$W^i(t) \leq \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\gamma(t_j) 1_{\{t_j > t\}}\right]}{\rho^\nu(t)} \tag{A.22}$$

because, if this inequality did not hold, given Assumption 2.1 part 3, the investor would derive higher utility by appropriately switching to the superreplicating strategy for γ (see Proposition A.2) which maintains nonnegative wealth and consuming more than $\gamma + e^i$ and, consequently, more than the equilibrium consumption C^i.

Each long lived investor who prefers more to less finances consumption net of endowments at its lowest possible cost, so each long lived investor i's equilibrium financial wealth must satisfy

$$W^i(t) - a^i(t) \leq \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\sum_{j=n+1}^{N+1} \rho^\nu(t_j)\big(\gamma(t_j) - g^i(t_j)\big)\right]}{\rho^\nu(t)} \quad \text{on } [t_n, t_{n+1})$$

because, if this inequality did not hold, the investor would derive higher utility by appropriately switching to the superreplicating strategy for γ (see Proposition A.2) and consuming more than $\gamma + e^i$ and, consequently, more than the equilibrium consumption C^i. Second, using this for long lived investors, (A.22) for short lived investors, plus market clearing for the assets implies aggregate financial wealth must be bounded:

$$\begin{aligned}
\bar{\pi}_S S^{\text{ex}}(t) - \sum_{i,\text{long lived}} a^i(t) &= \sum_i W^i(t) - \sum_{i,\text{long lived}} a^i(t) \\
&\leq I \times \operatorname{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\sum_{j=n+1}^{N+1} \rho^\nu(t_j)\big(\gamma(t_j) - \sum_{i,\text{long lived}} g^i(t_j)\big)\right]}{\rho^\nu(t)} \quad \text{on } [t_n, t_{n+1}). \qquad \text{(A.23)}
\end{aligned}$$

The following result is proved in the Section A.3.3.

Lemma A.3. *Let X and x be nonnegative progressively measurable stochastic processes with the properties*

1. $X(t_n) = 0$ *if* $t_n = T$.

2. $\rho^\nu(t_n \wedge T) X(t_n \wedge T) + \sum_{j=1}^{N+1} \rho^\nu(t_j) x(t_j) 1_{\{0 < t_j \leq t\}}$ *is a local martingale for all* $\nu \in \mathcal{V}$ *for all* n.

3. For each n, there exists a $\bar{\nu} \in \mathcal{V}$ such that $\rho^{\bar{\nu}}(t \wedge t_n)X(t \wedge t_n) + \sum_{j=1}^{N+1} \rho^{\bar{\nu}}(t_j)x(t_j)1_{\{0<t_j \leq t \wedge t_n\}}$ is a martingale on $[0, t_n \wedge T]$.

4. There exists a nonnegative progressively measurable process $\hat{\gamma}$ with

$$\sup_{\nu \in \mathcal{V}} E\left[\sum_{n=1}^{N+1} \rho^{\nu}(t_n)\hat{\gamma}(t_n)\right] < \infty.$$

and

$$X(t) \leq \text{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} \rho^{\nu}(t_j)\hat{\gamma}(t_j)1_{\{t_j>t\}}\right]}{\rho^{\nu}(t)}$$

then

$$X(t) = \text{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} \rho^{\nu}(t_j)x(t_j)1_{\{t_j>t\}}\right]}{\rho^{\nu}(t)}$$

and there exists a $\nu^* \in \mathcal{V}$ such that

$$X(t) = \frac{E_t\left[\sum_{j=1}^{N+1} \rho^{\nu^*}(t_j)x(t_j)1_{\{t_j>t\}}\right]}{\rho^{\nu^*}(t)}$$

Proof. See Appendix A.3.3. □

For any $\nu \in \mathcal{V}$ the process

$$\rho^{\nu}(t)\left(\bar{\pi}_S S^{\text{ex}}(t) - \sum_{i,\text{long lived}} a^i(t)\right) + \sum_{j=1}^{N+1} \rho^{\nu}(t_j)\left(\bar{\pi}_S D(t_j) - \sum_{i,\text{long lived}} g^i(t_j)\right) 1_{\{0<t_j \leq t\}}$$

is a nonnegative local martingale on $[0, t_n \wedge T]$ for all n. Moreover, Theorem 4.1 implies for each n, there exists a $\bar{\nu} \in \mathcal{V}$ such that the process

$$\rho^{\bar{\nu}}(t)\left(\bar{\pi}_S S^{\text{ex}}(t) - \sum_{i,\text{long lived}} a^i(t)\right) + \sum_{j=1}^{N+1} \rho^{\bar{\nu}}(t_j)\left(\bar{\pi}_S D(t_j) - \sum_{i,\text{long lived}} g^i(t_j)\right) 1_{\{0<t_j \leq t\}}$$

is a martingale on $[0, t_n \wedge T]$. In addition, our Assumption 2.1 part 3 and our definition of the wealth constraints imply $\bar{\pi}_S S^{\text{ex}}(t_n) - \sum_{i,\text{long lived}} a^i(t_n) = 0$ when $t_n = T$. The statements in the Proposition now follow by setting $X(t) = \bar{\pi}_S S^{\text{ex}}(t) - \sum_{i,\text{long lived}} a^i(t)$, $x(t_j) = \bar{\pi}_S D(t_j) - \sum_{i,\text{long lived}} g^i(t_j)$ and $\hat{\gamma}(t_j) = I \times \left(\gamma(t_j) - \sum_{i,\text{long lived}} g^i(t_j)\right)$ in Lemma A.3.

A.3.3 Auxiliary Results for Sections A.3 and A.3.2

Proof of Proposition A.3. First observe that on the event A (since $\mathcal{I}_{N+1} = \emptyset$ when $N < \infty$) described in the Proposition $t_{n-1} < T$, $t_n \leq T$, and $W^{i,\text{cum}}(t_n) - a^{i,\text{cum}}(t_n) \geq 0$. The last statement follows from $W^i(t) \geq a^i(t)$ so $W^i(t_n-) \geq a^i(t_n-)$. Since $W^{i,\text{cum}}(t_n) = W^i(t_n-)$ and $a^{i,\text{cum}}(t_n) = a^i(t_n-)$, it follows $W^{i,\text{cum}}(t_n) - a^{i,\text{cum}}(t_n) \geq 0$. For the event A described in the Proposition, the process $\rho^{\nu}(t)(W^i(t) + c^i(t_n)1_{\{t=t_n\}} - a^i(t)) - g^i(t_n)1_{\{t=t_n\}}$

is a local martingale for $t \in [t_{n-1}, t_n]$ for every $\nu \in \mathcal{V}$ (see Lemma A.1). The process is also nonnegative so it is a supermartingale. Thus for $t \in [t_{n-1}, t_n]$

$$(W^i(t) - a^i(t))1_{\{A\}} \geq \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)(W^{i,\mathrm{cum}}(t_n) - a^{i,cum}(t_n))1_{\{A\}}\right]}{\rho^\nu(t)}. \tag{A.24}$$

It follows that $\sup_{\nu \in \mathcal{V}} E[\rho^\nu(t_n)(W^{i,\mathrm{cum}}(t_n) - a^{i,cum}(t_n))1_{\{A\}}] < \infty$. Let $X = (W^{i,\mathrm{cum}}(t_n) - a^{i,cum}(t_n))1_{\{A\}} \geq 0$, $g = 0$, and $a = 0$ in Proposition A.1. This then yields a trading strategy and a wealth process, $\tilde{W}$ defined on $[0, t_n \wedge T]$ which has a terminal payoff $\tilde{W}(t_n \wedge T) \geq X$ and maintains nonnegative wealth. Moreover, for $t \in [t_{n-1}, t_n]$

$$\tilde{W}(t)1_{\{A\}} = \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t[\rho^\nu(t_n)(W^{i,\mathrm{cum}}(t_n) - a^{i,cum}(t_n))1_{\{A\}}]}{\rho^\nu(t)} \tag{A.25}$$

Let $W(t) = \tilde{W}(t) + a^i(t)$. Then $W(t) \geq a^i(t)$ and $W(t_n) \geq X$. Assumption 2.1 then implies $W^i(t)1_{\{A\}} \leq W(t)1_{\{A\}}$ for $t \in [t_{n-1}, t_n]$ otherwise it is feasible to shift to the strategy generating $W(t_n)$, invest the surplus wealth and consume more at a future date. Combined with (A.24) the statement of the Proposition follows.

□

Proof of Lemma A.2. We first show the assumptions of the lemma imply

$$X(\tau)1_{\{A\}} = \operatorname{essup}_\nu \frac{E_\tau\left[\rho^\nu(T)X(t_n)1_{\{A\}}\right]}{\rho^\nu(\tau)},$$

P-almost surely for all stopping times τ taking values in $[t_{n-1}, t_n]$. Proposition A.1 will then imply $X(t_{n-1})1_{\{A\}}$ is the lowest cost of replicating $X(t_n)1_{\{A\}}$ with a portfolio that maintains nonnegative wealth. Lemma A.4 (presented next) will prove the existence of a $\nu^* \in \mathcal{V}$ that makes $\rho^{\nu^*}(t)X(t)1_{\{A\}}$ a martingale on $[t_{n-1}, t_n]$.

The process

$$V^\Lambda(t) = \operatorname{essup}_{\nu \in \mathcal{V}} \frac{E_t\left[\rho^\nu(t_n)\Lambda 1_{\{A\}}\right]}{\rho^\nu(t)}$$

exists and has an RCLL modification for which $\rho^\nu(t)V^\Lambda(t)$ is an RCLL supermartingale for any $\nu \in \mathcal{V}$ (the proof is virtually identical to Karatzas and Shreve (1998, Theorem 5.6.5)). Choose a fixed stopping time τ taking values in $[t_{n-1}, t_n]$ and an arbitrary $\hat{\nu} \in \mathcal{V}$. The properties of the essential supremum (see Footnote 15) and the Monotone Convergence Theorem imply, for any $\epsilon > 0$, there is a $\nu^\epsilon \in \mathcal{V}$ such that[17]

$$E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(t_n)}{\rho^{\nu^\epsilon}(\tau)}\Lambda 1_{\{A\}}\right] > E\left[\rho^{\hat{\nu}}(\tau)V^\Lambda(\tau)1_{\{A\}}\right] - \frac{\epsilon}{2}.$$

[17]If we define

$$\tilde{V}^\Lambda(\tau) = \operatorname{essup}_\nu \frac{E_\tau[\rho^\nu(t_n)\Lambda 1_A]}{\rho^\nu(\tau)} \tag{A.26}$$

for stopping times $\tau \in [t_{n-1}, t_n]$ it may differ from $V^\Lambda(\tau)$ since, although they agree on constant stopping times, they are defined differently for stopping times which are not constant. However, we are using a right continuous modification of V^Λ so the arguments in Karatzas and Shreve (1998, Remark 5.6.7) can be used to show $\tilde{V}^\Lambda(\tau) = V^\Lambda(\tau)$ P almost surely.

Because $\rho^{\nu^\epsilon}(t)X(t)1_{\{A\}}$ is a nonnegative local martingale on $[\tau, t_n]$, there is an increasing sequence of stopping times $\tau \leq \tau_j \to t_n$ such that $\lim_{j\to\infty} P\{\tau_j = t_n\} = 1$ and such that the stopped process is a martingale on $[\tau, \tau_j]$.[18] So choosing j large enough,

$$E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(t_n)}{\rho^{\nu^\epsilon}(\tau)}\Lambda 1_{\{A\bigcap\{\tau_j=t_n\}\}}\right] + \epsilon > E\big[\rho^{\hat{\nu}}(\tau)V^\Lambda(\tau)1_{\{A\}}\big].$$

The supermartingale property of V^Λ implies

$$V^\Lambda(\tau)1_{\{A\}} \geq \frac{E_\tau\left[\rho^{\nu^\epsilon}(t_n)\Lambda 1_{\{A\bigcap\{\tau_j=t_n\}\}} + \rho^{\nu^\epsilon}(\tau_j)V^\Lambda(\tau_j)1_{\{A\bigcap\{\tau_j<t_n\}\}}\right]}{\rho^{\nu^\epsilon}(\tau)},$$

so we have

$$\epsilon > E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(\tau_j)}{\rho^{\nu^\epsilon}(\tau)}V^\Lambda(\tau_j)1_{\{A\bigcap\{\tau_j<t_n\}\}}\right].$$

Since $V^\Lambda \geq X$,

$$\epsilon > E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(\tau_j)}{\rho^{\nu^\epsilon}(\tau)}X(\tau_j)1_{\{A\bigcap\{\tau_j<t_n\}\}}\right].$$

Because τ_j reduces the local martingale $\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(t)}{\rho^{\nu^\epsilon}(\tau)}X(t)1_{\{A\}}$ on $t \geq \tau$, we have

$$\begin{aligned}
E\big[\rho^{\hat{\nu}}(\tau)X(\tau)1_{\{A\}}\big] &= E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(t_n)}{\rho^{\nu^\epsilon}(\tau)}X(t_n)1_{\{A\bigcap\{\tau_j=t_n\}\}}\right] + E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(\tau_j)}{\rho^{\nu^\epsilon}(\tau)}X(\tau_j)1_{\{A\bigcap\{\tau_j<t_n\}\}}\right] \\
&\leq E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^\epsilon}(t_n)}{\rho^{\nu^\epsilon}(\tau)}X(t_n)1_{\{A\}}\right] + \epsilon \\
&\leq E\left[\rho^{\hat{\nu}}(\tau)\,\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(\tau)}\right] + \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ we find

$$E\big[\rho^{\hat{\nu}}(\tau)X(\tau)1_{\{A\}}\big] \leq E\left[\rho^{\hat{\nu}}(\tau)\,\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(\tau)}\right].$$

On the other hand, the supermartingale property of $\rho^\nu(t)X(t)1_{\{A\}}$ implies

$$X(\tau)1_{\{A\}} \geq \mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(\tau)},$$

so we conclude

$$X(\tau)1_{\{A\}} = \mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(\tau)},$$

P-almost surely. The existence of a $\nu^* \in \mathcal{V}$ that makes $\rho^{\nu^*}(t)X(t)$ a martingale on $[t_{n-1}, t_n]$ follows from Lemma A.4 (presented next). □

[18] This follows from Doob's Maximal Inequality applied to the supermartingale $\frac{\rho^{\nu^\epsilon}(t)X(t)}{\rho^{\nu^\epsilon}(\tau)X(\tau)}$ on $[\tau, t_n]$ using the stopping times $\tau_j = \inf\{t \in [\tau, T] | \frac{\rho^{\nu^\epsilon}(t)X(t)}{\rho^{\nu^\epsilon}(\tau)X(\tau)} \geq j\} \wedge t_n$. Then $P\{\tau_j < t_n\} \leq \frac{1}{j}$.

Several of our results use the the following lemma which connects the martingale property to attaining the supremum.

Lemma A.4. *Let $A \in \mathcal{F}_{t_{n-1}} \bigcap \{t_{n-1} < T\}$ and let X be a nonnegative $\mathcal{F}_t$-adapted process for which $\rho^\nu X 1_{\{A\}}$ is a continuous local martingale on $[t_{n-1}, t_n]$ for all $\nu \in \mathcal{V}$, and assume $P(\rho^0(t_n)1_{\{A\}} > 0) = 1$ as in Assumption 3.2. Then*

$$X(\tau)1_{\{A\}} = \text{essup}_{\nu\in\mathcal{V}} \frac{E_\tau\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(\tau)} \tag{A.27}$$

for all stopping times τ taking values in $[t_{n-1}, t_n]$ if and only if there exists a $\nu^* \in \mathcal{V}$ for which $\rho^{\nu^*}(t)X(t)1_{\{A\}}$ is a martingale on $[t_{n-1}, t_n]$.

Proof. NECESSITY: Given (A.27), we explicitly construct a $\nu^* \in \mathcal{V}$ that makes the process $\rho^{\nu^*}(t)X(t)1_{\{A\}}$ a martingale. The ν^* we construct has the form

$$\nu^*(t) = \nu^0(t)1_{\{t\le t_{n-1}\}} + \sum_{m=1}^{\infty} \nu^m(t)1_{\{\tau^{m-1}<t\le\tau^m\}}, \tag{A.28}$$

where each $\nu^m \in \mathcal{V}$ and $\{\tau^m\}$ is an increasing sequence of stopping times with $\tau^0 = t_{n-1}$ and $\tau^m \uparrow t_n$ almost surely. The important properties of ν^* are that, for all m,

$$E\big[\rho^{\nu^*}(\tau^m)X(\tau^m)1_{\{A\}}\big] = E\big[\rho^{\nu^*}(t_{n-1})X(t_{n-1})1_{\{A\}}\big] \tag{A.29}$$

and

$$E\big[\rho^{\nu^*}(\tau^m)X(\tau^m)1_{\{A\bigcap\{\tau_m<T\}\}}\big] \le \epsilon_m \tag{A.30}$$

for a sequence $\{\epsilon_m\}$ of real numbers that decrease to zero as $m \to \infty$. Property (A.29) implies $\rho^{\nu^*}X1_{\{A\}}$ is a nonnegative local martingale and, consequently, a supermartingale on $[t_{n-1}, t_n]$. Property (A.30) and the Monotone Convergence Theorem additionally imply

$$\lim_{m\to\infty} E\big[\rho^{\nu^*}(t_n)X(t_n)1_{\{A\bigcap\{\tau_m=t_n\}\}}\big] = E\big[\rho^{\nu^*}(t_n)X(t_n)1_{\{A\}}\big] = E\big[\rho^{\nu^*}(t_{n-1})X(t_{n-1})1_{\{A\}}\big],$$

so $\rho^{\nu^*}X1_{\{A\}}$ must have constant expectation on $[t_{n-1}, t_n]$ and is therefore a martingale on $[t_{n-1}, t_n]$. We show later in the proof the ν^* we construct is in $\mathcal{V}$.

Start by choosing an arbitrary $\hat{\nu} \in \mathcal{V}$ and set $\nu^0(t) = \hat{\nu}(t)$. We now construct a ν^* of the form (A.28) using induction. Take as given a sequence of positive real numbers $\{\epsilon_m\}$ for which $\epsilon_m \downarrow 0$. We start by defining $\nu^1 \in \mathcal{V}$ and the stopping time τ^1. Using the properties of the essential supremum (per Footnote 15 applied to (A.27)), there is a $\nu^1 \in \mathcal{V}$ such that

$$E[\rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(t_n)}{\rho^{\nu^1}(t_{n-1})}X(t_n)1_{\{A\}}] + \frac{\epsilon_1}{2} \ge E[\rho^{\hat{\nu}}(t_{n-1})X(t_{n-1})1_{\{A\}}]. \tag{A.31}$$

To find τ^1, we first define the increasing sequence of stopping times by

$$\tau^1_j = \inf\{t \ge t_{n-1} | \rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(t)}{\rho^{\nu^1}(t_{n-1})}X(t)1_{\{A\}} \ge j\} \wedge t_n.$$

Because $\rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(t)}{\rho^{\nu^1}(t_{n-1})}X(t)1_{\{A\}}$ is a nonnegative local martingale (supermartingale) on $[t_{n-1}, t_n]$, Doob's Maximal Inequality implies

$$P\{\tau^1_j < t_n\} = P\left\{\sup_{\{t\in[t_{n-1},t_n]\}} \rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(t)}{\rho^{\nu^1}(t_{n-1})}X(t)1_{\{A\}} \ge n\right\} \le \frac{E[\rho^{\hat{\nu}}(t_{n-1})X(t_{n-1})1_{\{A\}}]}{j}$$

(Revuz and Yor, 1994, Theorem II.1.7). Consequently, $\tau^1_j \uparrow t_n$ almost surely, and the Monotone Convergence Theorem implies there is an j^1 satisfying both

$$P\{\tau^1_{j^1} < t_n\} \leq \frac{E[\rho^{\hat{\nu}}(t_{n-1})X(t_{n-1})1_{\{A\}}]}{j^1} < \frac{\epsilon^1}{2}$$

and

$$E\big[\rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(t_n)}{\rho^{\nu^1}(t_{n-1})}X(t_n)1_{\{A\bigcap\{\tau^1_{j^1}<T\}\}}\big] < \frac{\epsilon_1}{2}. \tag{A.32}$$

Define $\tau^1 \equiv \tau^1_{j^1}$, and let $\nu^*(t) = \nu^0(t)1_{\{t\leq t_{n-1}\}} + \nu^1(t)1_{\{t_{n-1}<t\leq\tau^1\}}$ on the set $\{0 \leq t \leq \tau^1\}$.

Before proceeding, we verify properties (A.29) and (A.30) for the ν^1 and τ^1 that define ν^* up to this point. Note the stopped process $\rho^{\nu^*}(t\wedge\tau^1)X(t\wedge\tau^1)1_{\{A\}}$ is a uniformly bounded and nonnegative local martingale on $[t_{n-1}, t_n]$, so it is a martingale by Lebesgue's Dominated Convergence Theorem. Therefore $E\big[\rho^{\nu^*}(\tau^1)X(\tau^1)1_{\{A\}}\big] = E\big[\rho^{\nu^*}(t_{n-1})X(t_{n-1})1_{\{A\}}\big]$, which verifies (A.29) for $m = 1$. Given this, we have

$$E\big[\rho^{\nu^*}(\tau^1)X(\tau^1)1_{\{A\bigcap\{\tau^1<t_n\}\}}\big]+E\big[\rho^{\nu^*}(t_n)X(t_n)1_{\{A\bigcap\{\tau^1=t_n\}\}}\big] = E\left[\rho^{\nu^*}(t_{n-1})X(t_{n-1})1_{\{A\}}\right]. \tag{A.33}$$

Subtracting (A.33) from (A.31) and using the identity $\rho^{\nu^*}(t) = \rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(t)}{\rho^{\nu^1}(t_{n-1})}$ we obtain the inequality

$$E[\rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(t_n)}{\rho^{\nu^1}(t_{n-1})}X(t_n)1_{\{A\bigcap\{\tau^1<t_n\}\}}]+\frac{\epsilon_1}{2}-E\big[\rho^{\hat{\nu}}(t_{n-1})\frac{\rho^{\nu^1}(\tau^1)}{\rho^{\nu^1}(t_{n-1})}X(\tau^1)1_{\{A\bigcap\{\tau^1<T\}\}}\big] \geq 0$$

Our choice of j^1 in (A.32) therefore implies

$$E\big[\rho^{\nu^*}(\tau^1)X(\tau^1)1_{\{A\bigcap\{\tau^1<t_n\}\}}\big] < \epsilon_1.$$

Thus our construction of ν^* up to this point also satisfies property (A.30) for $m = 1$.

We continue our construction of ν^* using induction. Let $k \geq 2$ be an integer, and suppose we have a ν^* of the form $\nu^*(t) = \nu^0(0) + \sum_{m=1}^{k-1}\nu^m(t)1_{\{\tau^{m-1}<t\leq\tau^m\}}$ on the set $\{0 \leq t \leq \tau^{k-1}\}$ for which the properties (A.29) and (A.30) hold for all m, $1 \leq m \leq k-1$, given the sequence $\{\epsilon_m\}$. We are assuming

$$X(\tau^k)1_{\{A\}} = \mathrm{essup}_{\nu\in\mathcal{V}}\,\frac{E_{\tau^k}\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(\tau^k)},$$

so (per Footnote 15) there is a $\nu^k \in \mathcal{V}$ so that

$$E\left[\rho^{\nu^*}(\tau^{k-1})\frac{\rho^{\nu^k}(t_n)}{\rho^{\nu^k}(\tau^{k-1})}X(t_n)1_{\{A\}}\right]+\frac{\epsilon_j}{2} \geq E\left[\rho^{\nu^*}(\tau^{k-1})X(\tau^{k-1})1_{\{A\}}\right] = E\left[\rho^{\hat{\nu}}(t_{n-1})X(t_{n-1})1_{\{A\}}\right]. \tag{A.34}$$

To select τ^k, we first define the increasing sequence of stopping times for $j > j^{k-1}$ by

$$\tau^k_j = \inf\left\{t\,\middle|\, t \geq \tau^{k-1} \text{ and } \rho^{\nu^*}(\tau^{k-1})\frac{\rho^{\nu^k}(t)}{\rho^{\nu^k}(\tau^{k-1})}X(t)1_{\{A\}} \geq j\right\} \wedge t_n.$$

Doob's Maximal inequality again implies $\tau_j^k \uparrow t_n$ almost surely. This and the Monotone Convergence Theorem implies there is an j^k satisfying both $P\{\tau_{j^k} < t_n\} \leq \frac{E\left[\rho^{\hat{\nu}}(t_{n-1})X(t_{n-1})1_{\{A\}}\right]}{j^k} < \frac{\epsilon_k}{2}$ and

$$E\left[\rho^{\nu^*}(\tau^{k-1})\frac{\rho^{\nu^k}(t_n)}{\rho^{\nu^k}(\tau^{k-1})}X(t_n)1_{\{\tau_{j^k}^k < T\}}\right] < \frac{\epsilon_k}{2}. \tag{A.35}$$

Define $\tau^k \equiv \tau_{j^k}^k$, and on the set $\{0 \leq t \leq \tau^k\}$ let $\nu^*(t) = \nu^1(0) + \sum_{m=1}^{j} \nu^m(t)1_{\{\tau^{m-1} < t \leq \tau^m\}}$.

We verify properties (A.29) and (A.30) for $m \leq k$. The stopped process $\rho^{\nu^*}(t \wedge \tau^k)X(t \wedge \tau^k)$ is a uniformly bounded nonnegative local martingale on $[t_{n-1}, t_n]$, so it is a martingale by Lebesgue's Dominated Convergence Theorem. This implies $E\left[\rho^{\nu^*}(\tau^k)X(\tau^k)1_{\{A\}}\right] = E\left[\rho^{\nu^*}(t_{n-1})X(t_{n-1})1_{\{A\}}\right]$, which verifies property (A.29) for $m = k$. The martingale property of the stopped process and the fact that property (A.29) holds for $m = k-1$ imply

$$E\left[\rho^{\nu^*}(t_{n-1})X(t_{n-1})1_{\{A\}}\right] = E\left[\rho^{\nu^*}(\tau^{k-1})X(\tau^{k-1})1_{\{A\}}\right] = E\left[\rho^{\nu^*}(\tau^{k-1})\frac{\rho^{\nu^k}(\tau^k)}{\rho^{\nu^k}(\tau^{k-1})}X(\tau^k)1_{\{A\}}\right]$$

$$= E\left[\rho^{\nu^*}(\tau^{k-1})\frac{\rho^{\nu^k}(\tau^k)}{\rho^{\nu^k}(\tau^{k-1})}X(\tau^k)1_{\{A\bigcap\{\tau^k < t_n\}\}}\right] + E\left[\rho^{\nu^*}(\tau^{k-1})\frac{\rho^{\nu^k}(t_n)}{\rho^{\nu^k}(\tau^{k-1})}X(t_n)1_{\{A\bigcap\{\tau^j = t_n\}\}}\right] \tag{A.36}$$

Subtracting (A.36) from (A.34) yields and applying (A.35), we obtain

$$E\left[\rho^{\nu^*}(\tau^k)X(\tau^k)1_{\{A\bigcap\{\tau^k < t_n\}\}}\right] = E\left[\rho^{\nu^*}(\tau^{k-1})\frac{\rho^{\nu^k}(\tau^k)}{\rho^{\nu^k}(\tau^{k-1})}X(\tau^k)1_{\{A\bigcap\{\tau^k < t_n\}\}}\right] < \epsilon_k,$$

which verifies property (A.30) for $m = k$.

Induction provides a sequence $\{\nu^m\}$ in $\mathcal{V}$ and an increasing sequence of stopping times $\{\tau^m\}$. Because $P(\tau^m < t_n) \downarrow 0$, the process $\nu^*(t) = \nu^0(t)1_{\{t \leq t_{n-1}\}} + \sum_{m=1}^{\infty} \nu^m(t)1_{\{\tau^{m-1} < t \leq \tau^m\}}$ is defined on $[0, t_n]$. Its construction makes it progressively measurable. To verify $\nu^* \in \mathcal{V}$, we must check the integrability condition $\int_{t_{n-1}}^{t_n} \|\nu^*(t)\|^2 dt < \infty$ almost surely. Note that $\rho^{\nu^*}(t_n) = \rho^{\nu^m}(t_n)$ on the event $\{A\bigcap\{\tau^m = t_n\}\}$, and $P(\rho^{\nu^m}(t_n)1_{\{A\}} > 0) = 1$ since $\nu_m \in \mathcal{V}$ and $P(\rho^0(t_n)1_{\{A\}} > 0) = 1$ (which is implied by Assumption 3.2). Therefore $P(\rho^{\nu^*}(t_n)1_{\{A\}} = 0) \leq P(A\bigcap\{\tau^m < t_n\}) \to 0$ as $m \to \infty$, so $P(\rho^{\nu^*}(t_n)1_{\{A\}} = 0) = 0$. The integrability condition then follows from Revuz and Yor (1994, IV.3.25)

SUFFICIENCY: Because $\rho^\nu X1_{\{A\}}$ is a nonnegative local martingale for all $\nu \in \mathcal{V}$, it is also a supermartingale. Therefore

$$\text{essup}_{\nu\in\mathcal{V}} \frac{E_{t_{n-1}}\left[\rho^\nu(t_n)X(t_n)1_{\{A\}}\right]}{\rho^\nu(t_{n-1})} \leq X(t_{n-1})1_{\{A\}}.$$

If there exists a ν^* that makes $\rho^{\nu^*}X$ a martingale, then we have $E\left[\rho^{\hat{\nu}}(t_{n-1})X(t_{n-1})1_{\{A\}}\right] = E[\rho^{\nu^*}(t_n)X(t_n)1_{\{A\}}] \leq E[\rho^{\hat{\nu}}(t_{n-1})\,\text{essup}_{\nu\in\mathcal{V}} \frac{\rho^\nu(t_n)}{\rho^\nu(t_{n-1})}X(t_n)1_{\{A\}}] \leq E\left[\rho^{\hat{\nu}}(t_{n-1})X(t_{n-1})1_{\{A\}}\right]$, which proves the statement. $\square$

Proof of Lemma A.3. The process

$$V^\Gamma(t) = \text{essup}_{\nu\in\mathcal{V}} \frac{E_t\left[\sum_{j=1}^{N+1} \rho^\nu(t_j)\gamma(\hat{t}_j)1_{\{t_j > t\}}\right]}{\rho^\nu(t)}$$

exists and has an RCLL modification for which $\rho^{\nu}(t)V^{\Gamma}(t)+\sum_{j=1}^{N+1}\rho^{\nu}(t_j)\hat{\gamma}(t_j)1_{\{t_j\le t\}}$ is an RCLL supermartingale for any $\nu\in\mathcal{V}$ (the proof is virtually identical to Karatzas and Shreve (1998, Theorem 5.6.5)). Choose a fixed stopping time τ and an arbitrary $\hat{\nu}\in\mathcal{V}$. The properties of the essential supremum (see Footnote 15) and the Monotone Convergence Theorem imply, for any $\epsilon>0$, there is a $\nu^{\epsilon}\in\mathcal{V}$ such that[19]

$$E\left[\rho^{\hat{\nu}}(\tau)\sum\frac{\rho^{\nu^{\epsilon}}(t_j)}{\rho^{\nu^{\epsilon}}(\tau)}\hat{\gamma}(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]>E\left[\rho^{\hat{\nu}}(\tau)V^{\Gamma}(\tau)\right]-\frac{\epsilon}{3}.$$

Choosing n^* large enough for all $n\ge n^*$,

$$E\left[\rho^{\hat{\nu}}(\tau)\sum_{j=1}^{n\wedge N+1}\frac{\rho^{\nu^{\epsilon}}(t_j)}{\rho^{\nu^{\epsilon}}(\tau)}\hat{\gamma}(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]+\frac{2\epsilon}{3}>E\big[\rho^{\hat{\nu}}(\tau)V^{\Gamma}(\tau)\big].$$

By assumption, for any n there exists a $\bar{\nu}$ such that $\rho^{\bar{\nu}}(t\wedge t_n)X(t\wedge t_n)+\sum\rho^{\bar{\nu}}(t_j)x(t_j)1_{\{t_j\le t\wedge t_n\}}$ is a martingale on $[0,t_n\wedge T]$. Define $\tau_m=\inf\{t|\rho^{\nu^{\epsilon}}(t)X(t)+\sum\rho^{\nu^{\epsilon}}(t_j)x(t_j)1_{\{t_j\le t\}}\ge m\}$ and $\nu^m(t)=\nu^{\epsilon}(t)1_{\{\tau_m\le t\}}+\bar{\nu}(t)1_{\{\tau_m>t\}}$. Then $\rho^{\nu^m}(t\wedge t_n)X(t\wedge t_n)+\sum\rho^{\nu^m}(t_j)x(t_j)1_{\{t_j\le t\wedge t_n\}}$ is a martingale and Fatou's Lemma implies

$$\liminf_{n\to\infty}E\left[\rho^{\hat{\nu}}(\tau)\sum_{j=1}^{n\wedge N+1}\frac{\rho^{\nu^m}(t_j)}{\rho^{\nu^m}(\tau)}\hat{\gamma}(t_j)1_{\{t_j>\tau\}}\right]\ge E\left[\rho^{\hat{\nu}}(\tau)\sum_{j=1}^{n\wedge N+1}\frac{\rho^{\nu^{\epsilon}}(t_j)}{\rho^{\nu^{\epsilon}}(\tau)}\hat{\gamma}(t_j)1_{\{t_j>\tau\}}\right] \tag{A.38}$$

Thus, by passing to a subsequence if necessary, there exists an m^* such that for all $m\ge m^*$

$$\begin{aligned}E\left[\rho^{\hat{\nu}}(\tau)\sum_{j=1}^{n\wedge N+1}\frac{\rho^{\nu^m}(t_j)}{\rho^{\nu^m}(\tau)}\hat{\gamma}(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]+\epsilon\ge E\left[\rho^{\hat{\nu}}(\tau)\sum_{j=1}^{n\wedge N+1}\frac{\rho^{\nu^{\epsilon}}(t_j)}{\rho^{\nu^{\epsilon}}(\tau)}\hat{\gamma}(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]+\frac{2\epsilon}{3}\\ >E\left[\rho^{\hat{\nu}}(\tau)V^{\Gamma}(\tau)\right].\end{aligned}$$

The process $\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^m}(t\wedge t_n)}{\rho^{\nu^m}(\tau)}X(t\wedge t_n)+\sum\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^m}(t_j)}{\rho^{\nu^m}(\tau)}x(t_j)1_{\{t_j\le t\wedge t_n\}}$ is a martingale on $[\tau,t_n\wedge T]$. The supermartingale property of V^{Γ} implies

$$V^{\Gamma}(\tau)\ge E_{\tau}\left[\sum_{j=1}^{n\wedge N+1}\frac{\rho^{\nu^m}(t_j)}{\rho^{\nu^m}(\tau)}\hat{\gamma}(t_j)\mathbf{1}_{\{t_j>\tau\}}+\frac{\rho^{\nu^m}(t_n)}{\rho^{\nu^m}(\tau)}V^{\Gamma}(t_n)1_{\{t_n<T\}}\right],$$

so we have

$$\epsilon>E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^m}(t_n)}{\rho^{\nu^m}(\tau)}V^{\Gamma}(t_n)1_{\{t_n<T\}}\right].$$

[19] If we define

$$\tilde{V}^{\Gamma}(\tau)=\operatorname{essup}_{\nu\in\mathcal{V}}\frac{E_{\tau}\left[\sum_{j=1}^{N+1}\rho^{\nu}(t_j)\hat{\gamma}(t_j)1_{\{t_j>\tau\}}\right]}{\rho^{\nu}(\tau)} \tag{A.37}$$

for stopping times $\tau\in[0,T]$ it may differ from $V^{\Gamma}(\tau)$ since, although they agree on constant stoppping times, they are defined differently for stopping times which are not constant. However, we are using a right continous modification of V^{Λ} so the arguments in Karatzas and Shreve (1998, Remark 5.6.7) can be used to show $\tilde{V}^{\Lambda}(\tau)=V^{\Lambda}(\tau)$ P almost surely.

By assumption $V^{\Gamma} \geq X$,

$$I\epsilon > E\left[\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^m}(t_n)}{\rho^{\nu^m}(\tau)}X(t_n)1_{\{t_n<T\}}\right]. \tag{A.39}$$

Because $\rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^m}(t\wedge t_n)}{\rho^{\nu^m}(\tau)}X(t\wedge t_n)+\sum \rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^m}(t_j)}{\rho^{\nu^m}(\tau)}x(t_j)\mathbf{1}_{\{t_j\leq t\wedge t_n\}}$ is a martingale on $[\tau, t_n\wedge T]$ and by assumption $X(t_n) = 0$ when $t_n = T$ we have

$$\begin{aligned}
E\big[\rho^{\hat{\nu}}(\tau)X(\tau)\big] &= E\left[\rho^{\hat{\nu}}(\tau)\sum_{j=1}^{n\wedge N+1}\frac{\rho^{\nu^m}(t_j)}{\rho^{\nu^m}(\tau)}x(t_j)1_{\{t_j>\tau\}} + \rho^{\hat{\nu}}(\tau)\frac{\rho^{\nu^m}(t_n)}{\rho^{\nu^m}(\tau)}X(t_n)1_{\{t_n<T\}}\right] \\
&\leq E\left[\rho^{\hat{\nu}}(\tau)\sum_{j=1}^{N+1}\frac{\rho^{\nu^m}(t_j)}{\rho^{\nu^m}(\tau)}x(t_j)1_{\{t_j>\tau\}}\right] + I\epsilon \\
&\leq E\left[\rho^{\hat{\nu}}(\tau)\,\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\sum_{j=1}^{N+1}\rho^{\nu}(t_j)x(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]}{\rho^{\nu}(\tau)}\right] + I\epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ we find

$$E\big[\rho^{\hat{\nu}}(\tau)X(\tau)\big] \leq E\left[\rho^{\hat{\nu}}(\tau)\,\mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\sum\rho^{\nu}(t_j)x(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]}{\rho^{\nu}(\tau)}\right].$$

The process $\rho^{\nu}(t)X(t) + \sum\rho^{\nu}(t_j)x(t_j)1_{\{t_j\leq t\}}$ is a nonnegative local martingale and thus a supermartingale so

$$X(\tau) \geq \mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\sum_{j=1}^{N+1}\rho^{\nu}(t_j)x(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]}{\rho^{\nu}(\tau)},$$

so we conclude

$$X(\tau) = \mathrm{essup}_{\nu\in\mathcal{V}}\frac{E_\tau\left[\sum_{j=1}^{N+1}\rho^{\nu}(t_j)x(t_j)\mathbf{1}_{\{t_j>\tau\}}\right]}{\rho^{\nu}(\tau)},$$

P-almost surely.

Finally, from (A.39) it also follows that we can construct a ρ^{ν^*} such that

$$\lim_{n\to\infty} E\left[\rho^{\nu^*}(t_n)X(t_n)\right] = 0$$

So

$$X(0) = E\left[\sum_{j=1}^{N+1}\rho^{\nu^*}(t_j)x(t_j)\right]$$

$\square$

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