

Optimal Dynamic Momentum Strategies

Kai Li¹ and Jun Liu²

¹ IFS, Southwestern University of Finance and Economics

² University of California San Diego

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Background and Main Results

Background

- Momentum has been extensively documented: Jegadeesh and Titman (1993), Moskowitz et al (2012), Asness et al (2013) ...
- Most strategies are myopic: big volatility, negative skewness and momentum crashes (Daniel and Moskowitz 2016)

Main results: explicitly solve the optimal momentum strategies for a long-run investor:

- It is not sufficient to only use standard momentum variable to explore the momentum effect;
- Need another model-induced momentum variable to hedge the momentum risk;
- The two momentums can jointly significantly forecast risk premium in the long run.

The Model

- (i) A risk-free asset with constant risk-free rate r ;
- (ii) The risky momentum asset price

$$\frac{dS_t}{S_t} = [\alpha m_t + (1 - \alpha)\mu + r]dt + \sigma dB_t, \quad (1)$$

- m_t : momentum variable, μ : constant average risk premium, α : the fraction of momentum.
- (iii) Moskowitz et al (2012): **“the past 12-month excess return of each instrument is a positive predictor of its future return.”**

$$m_t = \frac{1}{\tau} \int_{t-\tau}^t \left(\frac{dS_u}{S_u} - rdu \right). \quad (2)$$

- $\tau \geq 0$ is the ‘look-back period’ of the momentum.
- $\tau = 0$: i.i.d. return;
- $\alpha > 0$ momentum; $\alpha < 0$ reversal.
- AR process;

Return Characteristics

- The system has an almost surely continuously adapted **unique solution** $S > 0$ for a given \mathcal{F}_0 -measurable initial process $\varphi : \Omega \rightarrow C([- \tau, 0], R)$.
- The return process is **stationary** if and only if $-1 < \alpha < 1$.
- $$\ln S_T - \ln S_0 = \bar{\mu}_1 + e^{\frac{\alpha T}{\tau}} \left[\int_{-\tau}^{-\tau+T} \left[1 - e^{\frac{-\alpha(\tau+u)}{\tau}} \right] \frac{dS_u}{S_u} + \int_{-\tau+T}^0 \left(1 - e^{\frac{-\alpha T}{\tau}} \right) \frac{dS_u}{S_u} \right] + \sigma \int_0^T e^{\frac{\alpha(T-v)}{\tau}} dB_v.$$
- Different historical return paths, even with the same momentum, lead to different expected returns and Sharpe ratios.

The Optimization Problem

- The optimization problem of investors with expected utility over terminal wealth at time T and CRRA $\gamma > 0$ is given by

$$\sup_{(\phi_t)_{t \in [0, T]}} \mathbb{E}_0 \left[\frac{W_T^{1-\gamma}}{1-\gamma} \right].$$

- The price is inherently path-dependent, so **standard dynamic programming approach of Merton (1971) cannot be used.**
- We use the Cox-Huang approach:
 - originated from finance literature and applied to the non-Markov price models.
 - The market is complete \Rightarrow a unique state price density π_t .

The Optimal Dynamic Momentum Strategy

- When $0 \leq T \leq \tau$, the optimal portfolio is given by

$$\begin{aligned}\phi_0^* &= \phi_0^M + \phi_0^H, \\ \phi_0^M &= \frac{\alpha}{\gamma\sigma^2} \int_{-\tau}^0 \frac{1}{\tau} \left(\frac{dS_v}{S_v} - r dv \right) + \frac{(1-\alpha)\mu}{\gamma\sigma^2}, \\ \phi_0^H &= \frac{\alpha}{\gamma\sigma^2} \int_{-\tau}^0 (1-\gamma)\omega_v \left(\frac{dS_v}{S_v} - r dv \right) + C_1,\end{aligned}\tag{3}$$

where

$$\omega_v = \begin{cases} \int_{-\tau}^v \hat{\omega}_u du, & v \in [-\tau, -\tau + T], \\ \int_{-\tau}^{-\tau+T} \hat{\omega}_u du, & v \in [-\tau + T, 0], \end{cases} \quad \text{and } \hat{\omega}_u > 0.$$

- For large γ , ϕ_0^H is (to the leading order of $1/\gamma$) given by

$$\phi_0^H = \frac{-\alpha^2}{\gamma\sigma^2} \left\{ \int_{-\tau}^{-\tau+T} \frac{v+\tau}{\tau^2} \left(\frac{dS_v}{S_v} - r dv \right) + \int_{-\tau+T}^0 \frac{T}{\tau^2} \left(\frac{dS_v}{S_v} - r dv \right) + C_0 \right\}$$

Path Dependence

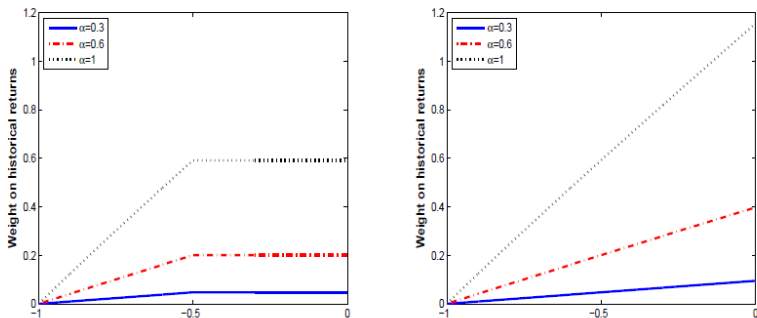


Figure: The weight ω_v on historical instantaneous excess return $dS_v/S_v - r dv$ for $v \in [-\tau, 0]$ in the hedging demand ϕ_0^H with horizon (a) $T = \tau/2$ ($< \tau$) and (b) $T = \tau$ respectively.

Path Dependence — Portfolio Dynamics

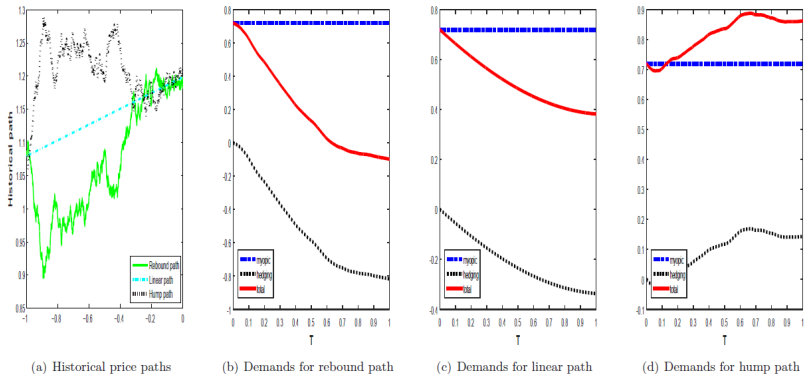


Figure: Panel (a): two typical stochastic historical paths with a rebound shape and a hump shape respectively a linear path. The three paths have the same momentum with the same beginning price and end price. The optimal portfolios are plotted against T in Panels (b), (c) and (d) for the rebound path, the linear path and the hump-shaped path respectively.

Horizon Dependence

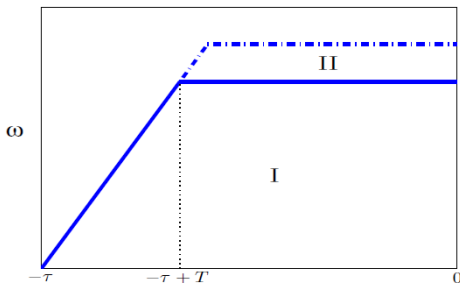


Figure: This figure illustrates the weights of the hedging demand on historical instantaneous excess returns. For an investment horizon T , area I is the weights on historical returns during $[-\tau, 0]$. When T increases, the weights increase by the amount illustrated by area II . The change in the weighted average can be positive or negative depending on historical return paths during $[-\tau + T, 0]$. This leads many intervals of increases and decreases in the portfolio weight as a function of horizon.

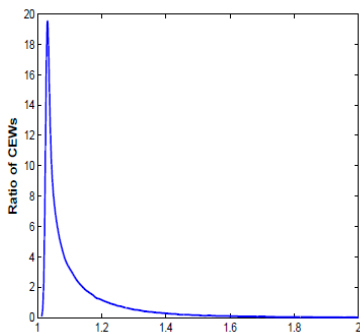
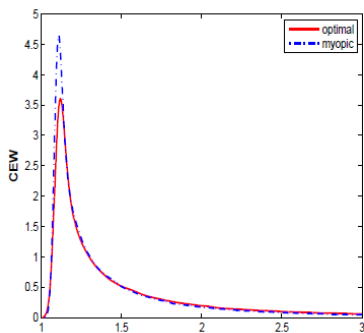


Figure: Distributions of the certainty equivalent wealths (CEWs) of the optimal and myopic strategies, and the ratio of their CEWs. The results are based on 10000 simulated historical paths generated from model.

$$X_{max} = \max_{t', t'' \in [-\tau, 0]} \left\{ (\ln S_{t'} - \ln S_{t''})^2 \right\} = \left(\max_{t \in [-\tau, 0]} \{\ln S_t\} - \min_{t \in [-\tau, 0]} \{\ln S_t\} \right)^2.$$

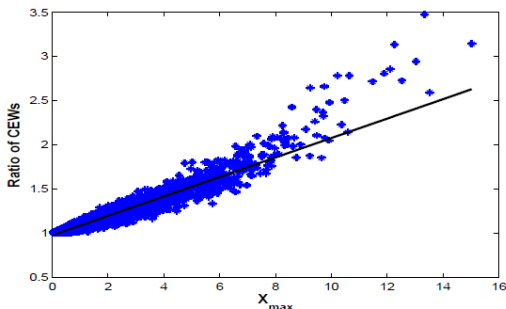


Figure: The results are based on 10000 simulated historical paths generated from model (1)-(2). The black solid line illustrates the linear regression $X_{max} = 0.98 + 0.11 \times \text{ratios of CEWs} + \epsilon$.

Empirical Results — Portfolio performance

Panel A: Parameter estimates (%)

α	μ	r	σ
22.7	4.9	3.6	16.4

Panel B: Portfolio weight ϕ^*

Horizon	1-Month	3-Month	6-Month	9-Month	12-Month
Mean	0.37	0.48	0.45	0.43	0.41
Std	0.32	0.27	0.28	0.28	0.29
Max	1.94	2.21	2.28	2.34	2.39
Min	-1.73	-1.21	-1.26	-1.30	-1.38

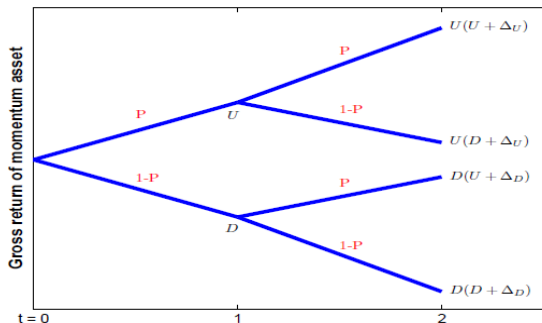
Panel C: Portfolio return and regression of $r^{T,*} - r^{T,M} = a + bX_{max} + \epsilon$

T	Mean (%)			Standard deviation (%)			Sharpe ratio (%)			b 's t -stat
	opt	myo	pas	opt	myo	pas	opt	myo	pas	
3m	7.14	6.79	8.52	10.5	11.4	18.8	33.3	27.7	26.0	(9.6)
6m	7.03	6.79	8.52	9.82	11.4	18.8	34.5	27.7	26.0	(11.1)
9m	6.96	6.79	8.52	9.66	11.4	18.8	34.4	27.7	26.0	(15.6)
12m	6.90	6.79	8.52	9.06	11.4	18.8	36.0	27.7	26.0	(19.1)

Empirical Results — Excess return prediction

T	Panel A: $r_{t:t+T} = a + bm_t + \epsilon_{t+1}$			Panel B: $r_{t:t+T} = a + bm_{T,t}^H + \epsilon_{t+1}$			
	b	R^2 (%)	R_{OS}^2 (%)	b	R^2 (%)	R_{OS}^2 (%)	
3m	0.37 (1.05)	0.44	-0.41	0.46 (1.35)	0.69	-0.23	
6m	0.54 (0.87)	0.45	-0.05	0.82 (1.46)	1.14	0.62	
9m	0.52 (0.62)	0.27	-0.15	0.92 (1.32)	0.97	0.59	
12m	-0.06 (-0.05)	0.00	-0.54	0.58 (0.63)	0.28	-0.30	
Panel C: $r_{t:t+T} = a + b_1m_t + b_2m_{T,t}^H + \epsilon_{t+1}$				b_1	b_2	R^2 (%)	R_{OS}^2 (%)
3m				-2.40 (-1.45)	2.77 (1.72)	1.34	-0.30
6m				-2.91 (-2.15)	3.46 (2.83)	2.40	1.57
9m				-2.72 (-1.85)	3.25 (2.97)	2.04	1.48
12m				-3.57 (-2.47)	3.51 (3.69)	1.87	1.21

An Illustrative Example of Reversing Momentum



- Two assets: momentum asset and riskless asset with constant gross return R_f .
- Momentum: $\Delta_D < 0 < \Delta_U$ and hence past-dependent.
- Constant conditional volatility at each node of the tree.

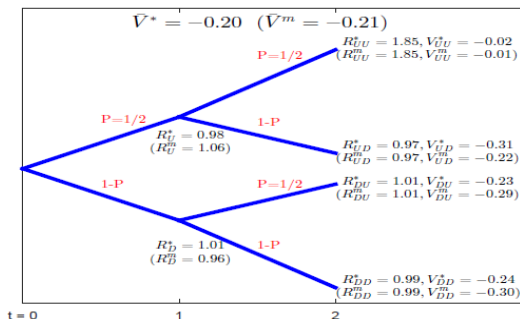
An Illustrative Example of Reversing Momentum

Expected CRRA Utility over Terminal Wealth

$$\begin{aligned} \max_{\phi_0, \phi_1} \mathbb{E}_0 \left[\frac{\tilde{W}_2^{1-\gamma}}{1-\gamma} \right] &= \max_{\phi_0, \phi_1} \mathbb{E}_0 \left[\frac{(W_0 \tilde{R}_1^P \tilde{R}_2^P)^{1-\gamma}}{1-\gamma} \right] \\ &= \max_{\phi_0} \mathbb{E}_0 \left[\frac{W_0^{1-\gamma}}{1-\gamma} (\tilde{R}_1^P)^{1-\gamma} \max_{\phi_1} \mathbb{E}_1 [(\tilde{R}_2^P)^{1-\gamma}] \right] \\ &= \max_{\phi_0} \mathbb{E}_0 \left[\frac{W_0^{1-\gamma}}{1-\gamma} (\tilde{R}_1^P)^{1-\gamma} \mathbb{E}_1 [(\tilde{R}_2^*)^{1-\gamma}] \right] \\ &= \max_{\phi_0} \mathbb{E}_0 \left[\frac{W_0^{1-\gamma}}{1-\gamma} (\tilde{R}_1^P)^{1-\gamma} \tilde{\zeta} \right] \end{aligned}$$

- Define $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \tilde{\zeta} \Rightarrow \max_{\phi_0} \mathbb{E}^* \left[\frac{W_0^{1-\gamma}}{1-\gamma} (\tilde{R}_1^P)^{1-\gamma} \right]$.
- $P^* < P$, and decreases as $\Delta = \Delta_U - \Delta_D$ increases.
- $\Delta \uparrow \Rightarrow \mathbb{E}^*[\tilde{R}_1] = P^*U + (1 - P^*)D < R_f \Leftrightarrow \phi_0 < 0$.

An Illustrative Example of Reversing Momentum



- The optimal demand $\phi_0^* = -0.05 < 0$, while the myopic demand $\phi_0^m = 0.13 > 0$.
- $\Delta_{U,D} = 0 \Rightarrow$ a standard i.i.d return process of risky asset $\Rightarrow P^* = P \Rightarrow$ always takes long position in the risky asset.

A New General Methodology

When $(n-1)\tau \leq T-t \leq n\tau$, $n=1, 2, \dots$, the optimal wealth fraction invested in the risky asset is given by $\phi_0^* = \phi_0^M + \phi_0^H$, where

$$\phi_0^H = A_{11,0}^{(n)} \ln S_0 + \sum_{i=2}^{2^n-1} A_{i1,0}^{(n)} \tilde{B}_{i,0}^{(n)} + A_{1,0}^{(n)}, \quad (4)$$

where $A^{(n)}$'s are governed by ODEs, and $\tilde{B}_{i,t}^{(n)}$'s are governed by random ODEs.

- the original state variable cannot constitute a sufficient statistic of the optimal portfolio.
- we solve the problem by introducing new state variables.
- the first to reduce the infinite-dimensional non-Markovian system to finite dimensional ODE systems.
- **many potential applications** for important problems in economics and finance.