Herding and Bank Runs

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Abstract

Traditional models of bank runs do not allow for herding effects, because in these models withdrawal decisions are assumed to be made simultaneously. I extend the banking model to allow a depositor to choose his withdrawal time. When he withdraws depends on his consumption type (patient or impatient), his private, noisy signal about the quality of the bank's portfolio, and the withdrawal histories of the other depositors. Some of these runs are efficient in that the bank is liquidated before the portfolio worsens. Others are not efficient; these are cases in which the herd is misled.

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1 Introduction

In the classic bank-runs model of Diamond and Dybvig (1983), individual withdrawal decisions are made simultaneously. The lack of detailed dynamics of withdrawals makes it difficult to explain some observed features of bank runs. In reality, at least some withdrawals are based on the information about the previous withdrawals of others.[1] During the 1994-1995 Argentine banking crisis, large depositors were responsible for most of the deposit outflows at the beginning of the crisis. Small depositors began to make substantial withdrawals two months later.[2] In their analysis on the runs on Turkish special finance houses (SFHs)[3] in 2001, Starr and Yilmaz (2007, p1114) find that depositors made sequential withdrawals influenced by the history of the withdrawals of others. The authors argue that the "increased withdrawals by moderate-size account holders tended to boost withdrawals by [their] small counterparts, suggesting that the latter viewed the former as informative with respect to the SFH's financial condition."

In the present paper, I build a model in which the timing of individual withdrawals is determined by the depositor's consumption type (*patient*, which means he does not need to consume early, or *impatient*, which means he needs to consume early), his noisy signal about the quality of the bank's portfolio, and the withdrawal history of other depositors. The signals are received in an exogenously determined sequence, but the timing of withdrawal is determined endogenously.[4] Because a depositor's simple withdraw-or-not action does not reveal perfectly to others the pair of private signals that he receives, other depositors can only imperfectly extract the depositor's private signals from his action. They update their beliefs about the quality of the bank's portfolio accordingly.

This paper does not focus on the panic-based bank runs of Diamond and Dybvig (1983). (See also Peck and Shell, 2003.) I focus instead on bank runs that occur as a

[1]Brunnermeier (2001, p. 214) says that "Although withdrawals by deposit holders occur sequentially in reality, the literature typically models bank runs as a simultaneous move game."

[2]See Schumacher (2000).

[3]Special financial houses are like commercial banks, but their deposits are not insured.

[4]Chamley and Gale (1994) and Gul and Lundholm (1995) were the first to introduce models of herding in investment decisions with endogenous timing. Such a setup has not been applied to bank deposit game, where payoff externality is important.

result of depositors trying to extract information about bank portfolio quality from the withdrawal histories of others. Because signals about the fundamentals are imperfect, and because signal extraction from the observed withdrawal history is also imperfect, a bank run can occur when the bank fundamentals are strong. In particular, it can occur when "too many" depositors receive early liquidity shocks. A bank run due to imperfect signal extraction only occurs in a setting with non-simultaneous withdrawal decisions. Bank runs in this sense are not purely fundamental-based.[5,6]

I show that there is a perfect Bayesian equilibrium in which a depositor withdraws if his expected utility is below his threshold level, and otherwise he waits. A depositor's expected utility depends upon his beliefs about the quality of the bank's portfolio. These beliefs are updated recursively by the observed withdrawal history of the other depositors. If a depositor's beliefs are in an intermediate range, he follows his private signals: If he is impatient or the portfolio signal is unfavorable, he withdraws; otherwise he waits. A bank run occurs as a result of a herd of withdrawals when all depositors withdraw due to unfavorable signals and/or unfavorable observations on withdrawals. If a depositor's belief becomes sufficiently favorable, the private signal he receives will not be decisive: The depositor waits to withdraw unless he is impatient. In this case, his private signal will not be revealed through his withdrawal behavior, so his withdrawal behavior does not affect others' beliefs or their expected utilities. A "no-bank-run" regime thus takes place as a result of a "herd of non-withdrawals."

Compared with herding in investment decisions (Banerjee, 1992; Bikhchandani et al., 1992; and more recently Chari and Kehoe, 2003, 2004), herding in bank runs has some special features that lead to interesting results. The most important difference lies in payoff externality. In the banking setup, a depositor's payoff depends not only on his own actions but also on the actions of others, in particular, whether a bank run occurs or not. So payoff externality adds additional uncertainty to a depositor's payoff. However, this payoff uncertainty is not necessarily bad, because a run can force the bank to liquidate

[5] See Allen and Gale (1994, 1998), Gorton (1988), and Calomiris and Gorton (1991), among others, for theoretic models and empirical evidence on fundamental-based runs.

[6] Goldstein and Pauzner (2005) construct a model in which depositors receive i.i.d. signals on fundamentals and determine whether to run on the bank simultaneously.

assets before low portfolio returns are actually realized, that is, before a higher welfare loss is incurred.

The present paper addresses a result that is paid little attention to in the traditional herding literature: One's expected utility is not necessarily monotone in one's own belief. Because we want to liquidate the bank early if its portfolio performs poorly and keep it operating if it performs well, information about the bank's portfolio is valuable. Although a more favorable belief makes a depositor more confident in the quality of the bank's portfolio, he knows it is also more likely to lead to a herd of non-withdrawals in which no more information will be made available in the future. Expected utility might not be increasing in the probability that the portfolio return is high, because a non-withdrawal herd reveals no further information about the bank's portfolio. Combined with payoff externality, the non-monotonicity can result in multiple threshold beliefs: A depositor can withdraw with a relatively favorable belief about the bank's portfolio performance because he expects he will not be able to accumulate sufficient information, but the possible bank run will hurt his chance of getting paid; However, if he has a less favorable belief and he expects more information to be revealed, he could prefer to wait because the gain from more information outweighs his probable loss in a bank run; If he has an unfavorable belief, the dim prospect of the bank's portfolio and the fear of loss in a bank run can dominate the incentive to wait.

There is literature on bank runs that is related to this paper.[7] Chen (1999) explains contagious bank runs using an information externality in the simplest two-stage game. Banks' portfolio returns are correlated. Bank runs are contagious because depositors infer negative information about their own banks' portfolio return from the observation of runs on other banks. Compared with Chen's work, my paper has a more general setup that can explain a bank run, as well as no run, for a single bank as a result of herd behavior.

[7]Yorulmazer (2003) sets up a similar model in which depositors receive signals about the portfolio returns in sequence in an attempt to explain herding runs. There are two major differences between his work and mine. First, the order of withdrawals in his model is determined exogenously as in Bikhchandani et al. (1992). The timing of withdrawals is endogeously determined in my paper. Second, Yorulmazer's analysis is focused on the case in which consumption types of the depositors are publicly observed. In my paper, consumption types are only privately observed.

Chari and Jagannathan (1988) analyze an economy in which the uninformed depositors infer information about productivity by observing the aggregate withdrawals rate. There is a rational expectation equilibrium in the model that allows for bank runs. However, Chari and Jagannathan use a static equilibrium concept, whereas my paper emphasizes the dynamics in the withdrawal process.

The remainder of this paper is organized as follows: The model is introduced in section 2. In section 3, I describe the equilibrium for a demand deposit contract. A perfect Bayesian Nash equilibrium is shown to exist. In section 4, I discuss the properties of the equilibrium. The final section offers some concluding observations.

2 Model Setup

Time: There are three time periods, index by $t = 0, 1, 2$. Period 0 is a planning period. Period 1 is divided into $N + 1$ stages, where N is a finite integer.

Depositor's endowments and preferences: The depositor's endowments and preferences are essentially as in Diamond and Dybvig (1983). There is a measure one of depositors. Each depositor is endowed with one unit of the consumption good in period 0, and nothing in periods 1 and 2. Depositors can store the consumption good at no cost in any period. Each depositor has probability α to become impatient in period 1. Impatient depositors derive utility only from consumption in period 1. Their utility is described by $u(C_1)$, where C_1 is the consumption received in period 1. The rest are patient depositors, who consume in the last period. If a patient depositor withdraws consumption in period 1, he can store it and consume it in period 2. Thus, a patient depositor's utility is described by $u(C_1 + C_2)$, where C_2 is the consumption received in period 2. The utility function is strictly increasing, strictly concave, and is normalized so $u(0) = 0$. The coefficient of relative risk aversion (CRRA), $-xu''(x)/u'(x)$, is greater than 1 for $x \geq 1$. Whether a depositor is patient or impatient is his private information and is revealed to the individual depositor at some stage in period 1. By the law of large numbers, α is also the fraction of depositors in the population who are impatient.

The bank and its technology: The bank behaves competitively. In addition to a costless storage technology, the bank can also invest in a productive asset. The investment in production can be made only in the initial period. Production is risky. One unit of consumption good invested in period 0 yields R units in period 2. $R = \overline{R} > 1$ with probability p_0, and $R = \underline{R} \leq 1$ with probability $1 - p_0$. The asset can be liquidated in period 1 with return of 1.[8] Either all or none must be liquidated. The productive asset can therefore be taken as an "indivisible good." An individual depositor cannot acquire the asset directly.

The contract: For convenience, I assume that if a depositor decides to deposit at the bank, the minimum amount of the deposit is one unit of consumption good. A competitive bank offers a simple demand deposit contract that describes the amount of consumption goods paid to the depositors who withdraw in periods 1 and 2, c_1 and c_2, respectively, where c_1 is independent of the stochastic asset return and c_2 is state contingent. The bank pays c_1 to the depositors at $t = 1$ until it is out of funds. If the amount of consumption good in storage cannot meet the withdrawal demand, the bank has to liquidate assets. The bank distributes the remaining resource (plus or minus the return on the portfolio) equally among the depositors who wait until the last period. Denote the fraction of deposits that the bank keeps in storage by λ and the fraction of depositors who withdraw in period 1 by β $(0 \leq \beta \leq 1)$.[9] The payment to a depositor who withdraws in period 2 is given by

$$c_2 = \begin{cases} \dfrac{\lambda - \beta c_1 + (1 - \lambda)R}{1 - \beta} & \text{if } \beta c_1 \leq \lambda \\ \dfrac{1 - \beta c_1}{1 - \beta} & \text{if } \lambda < \beta c_1 \leq 1 \\ 0 & \text{if } \beta c_1 > 1. \end{cases}$$

Because at least α of the depositors need to consume in period 1, λ must be at least αc_1. If the bank cannot meet its payment requirements in period 1, the bank fails. The bank does not liquidate the assets unless it is forced to do.

[8] Here, the liquidation value is set to be 1 for convenience. All results remain if the liquidation value is between $\underline{R}$ and $\overline{R}$. Setting the liquidation value higher than or equal to $\underline{R}$ captures the idea that weak banks should be liquidated early to avoid future losses.

[9] Later in the present paper, I will confine attention to symmetric equilibria in which β equals either α when there is no bank run or 1 when there is a bank run.

Withdrawal stages and information: In each of the first N stages of period 1, only one depositor is informed of his consumption type. Information about his type is precise. He also receives a noisy, private signal about the return of the bank's portfolio. Let S_n denote the signal about the bank's return obtained by the depositor who is informed at stage n. The signal is accurate with probability q, $q > 0.5$. That is,

$$\Pr(S_n = H | R = \overline{R}) = \Pr(S_n = L | R = \underline{R}) = q,$$

where H and L are high and low returns, respectively. The probability of receiving an accurate signal is q. After receiving a signal, the depositor updates his belief about portfolio returns by Bayes' rule. The common initial prior is p_0. At stage $N+1$, depositors who have not received signals are informed of their consumption types but not about asset returns. An impatient depositor has to consume at the stage when his consumption type is revealed to him.

Timing of the banking game:

The timeline of the banking game can be summarized as follows:

Period 0:	Bank announces the contract.
	Depositors make deposit decision.
Period 1:	
Stage n (1 through N):	One depositor receives signals about his consumption type and about asset returns.
	He decides whether to withdraw or not.
	Remaining depositors decide whether to withdraw or not.
Stage $N+1$:	Consumption types are revealed to those who have not been informed.
	Depositors decide whether to withdraw or not.
Period 2:	Bank allocates the remaining resources to those who have not withdrawn in period 1.

Depositors are equally likely to be informed at each stage. Because N is small compared with the continuum depositors, the probability of getting information in the first N stages is zero. Depositors do not communicate with each other about the signals they receive. However, a depositor's withdrawal action is observed by all others.[10,11] Once a depositor withdraws, he cannot reverse his decision. But if a depositor chooses to wait, he can withdraw at a later stage. The final deadline for depositors to withdraw in period 1 is stage $N+1$. Depositors are not allowed to change decisions after observing other depositors' decisions at stage $N+1$.

We can divide depositors into four types at each of the first N stages. The first type is those who already have withdrawn their deposits from the bank. These are inactive depositors who have no more decisions to make. The newly informed depositor who receives signals in the current stage is of the second type. The third type consists of the depositors who were informed at previous stages but have been waiting (call them already informed depositors). The remaining type is the uninformed. Let $\mathcal{T} = \{I, A, U\}$ be the set of types of active depositors at a stage, where I, A, and U represent newly informed, already informed, and uninformed depositors, respectively. The bank does not receive private information about asset returns. It is in the same position as an uninformed depositor in terms of information.

A finite number of stages is necessary because it imposes a deadline for the depositors to make decisions in period 1, so expected utility can be calculated by backward induction given the beliefs. The specification of a continuum of depositors greatly simplifies calculation and presentation. In contrast, consider a model that has a finite number of depositors. There exists at least one perfect Bayesian equilibrium in a finite game in which the beliefs and actions of a depositor are affected by the actions of others. Because each depositor has an atomic share at the bank, every withdrawal has a significant impact on the amount of remaining resource at the bank, which complicates the calculation of expected utility. The description of the equilibrium will be dependent on the parameters

[10] I consider the limit of large finite economies. Individual withdrawals are observable, as in an economy with a large number of depositors, yet the effects on the total resources is negligible. Also see footnote 17.

[11] I assume that if a patient depositor decides to withdraw, he imitates an impatient depositor and withdraws all of his deposits. Otherwise, the bank can distinguish the depositor's true type and can write a contract to refuse to pay him in period 1. Therefore, the depositor's actions are discrete. Lee (1993, 1998) shows that with exogenous timing of actions, herds cannot occur if the actions are continuous. Whereas Chari and Kehoe (2004) show that with endogenous timing of actions, herds occur even if the actions are continuous. Including continuous actions in the model would be an interesting extension.

of the economy, and there will be many more cases to discuss.[12]

In the following section, I will show that there exists a perfect Bayesian equilibrium in period 1 given a demand deposit contract. I will discuss the bank's choice of contract at the end of section 4.

3 Deposit Game

In Diamond and Dybvig (1983), a demand deposit contract that offers $c_1 > 1$ allows for a panic-based bank run in the deposit game. For convenience, although panic-based runs are allowed in the setup of the present paper, I do not consider them. A bank run occurs in my model due to private information about the asset returns and imperfect extraction of this information from observing the history of withdrawals.

Depositors observe the total number of withdrawals at each stage. Let X_n denote the total number of withdrawals at stage n. The public history of withdrawals records the total number of withdrawals at each stage up to stage n. A depositor's private history as of stage n differs from the public history only if he has received signals at stage m, $m \leq n$. A depositor's belief at stage n is a function that maps his private history into the probability that the asset returns are high. A depositor's strategy at stage n is a function that maps his private history into a zero-one withdrawal decision.

To simplify the notation, let x_n^τ and p_n^τ, $\tau = I$, A, U, denote the strategy and posterior belief that the asset return is high, respectively, of a type τ depositor at stage n. Let $x_n^\tau = 0$ represent a depositor's decision to wait at stage n and let $x_n^\tau = 1$ represent the decision to withdraw.

The equilibrium concept here is perfect Bayesian equilibrium. In particular, I consider a symmetric pure strategy perfect Bayesian equilibrium in which depositors with the same history take the same action at each stage.

For a contract that offers $c_1 < 1$, there does not exist a symmetric pure strategy run equilibrium, because given that all others withdraw from the bank, an individual

[12] In the appendix, I present a simple example of a two-stage, two-depositor economy. Similar results are obtained in the example.

depositor prefers to wait to get all the remaining resources, which is expected to be an infinite amount. Not withdrawing before stage $N+1$ is a patient depositor's dominant strategy regardless of all other depositors' actions and signals. Because an uninformed depositor never infers any information about asset returns from the actions of the newly informed depositors, a bank run does not occur. The analysis in the rest of section 3 is based on the assumption that $c_1 \geq 1$.

3.1 Bayesian Updates

A newly informed depositor updates his beliefs about the asset returns being high by the productivity signal he receives. Let $P_H(p)$ and $P_L(p)$ be the posterior probabilities that the asset return is high if a high or a low signal is received, respectively, given the prior of p. From Bayes' rule, we have

$$P_H(p) = \frac{pq}{pq + (1-p)(1-q)},$$

and

$$P_L(p) = \frac{p(1-q)}{p(1-q) + (1-p)q}.$$

Given that $q > 0.5$, we have $p \leq P_H(p) \leq 1$ and $0 \leq P_L(p) \leq p$ for $p \in [0,1]$. $P_H(p)$ and $P_L(p)$ are strictly increasing in p. Note that because the signals are of the same quality, we have $P_H(P_L(p)) = P_L(P_H(p)) = p$.

Other depositors update their beliefs about the asset returns being high by observing the actions taken by the newly informed depositor. If other depositors think that the newly informed depositor does not make decisions according to his signal about productivity, that is, he withdraws if and only if he is of an impatient type, other depositors do not change their beliefs, because the action of the newly informed depositor carries no information about the productivity. Suppose, alternatively, that other depositors believe that the newly informed depositor waits if and only if a high signal is received and he is patient. If the newly informed depositor waits, other depositors update their beliefs by P_H at stage n. However, if the newly informed depositor withdraws, others depositors update their beliefs by

$$P_{\widetilde{L}}(p) = \frac{p\,(1-q+\alpha q)}{\alpha + (1-\alpha)\,[p(1-q)+(1-p)\,q]}.$$

Here $P_{\widetilde{L}}\,(p)$ denotes posterior probability where the probability of observing an impatient depositor is taken into account, given the prior of p. For $p \in [0,1]$, we have $0 \leq P_L\,(p) \leq P_{\widetilde{L}}(p) \leq p$. Note that $P_H\left(P_{\widetilde{L}}\,(p)\right) = P_{\widetilde{L}}(P_H\,(p))$. It follows that the effects of a sequence of observed actions on the prior can be summarized by the number of non-withdrawals and the number of withdrawals in the sequence.

3.2 A Perfect Bayesian Equilibrium

3.2.1 Beliefs and strategies

To simplify the notation, let $u_1 = u\,(c_1)$, $\overline{u}_2 = u\left(\frac{\lambda-\alpha c_1+(1-\lambda)\overline{R}}{1-\alpha}\right)$, and $\underline{u}_2 = u\left(\frac{\lambda-\alpha c_1+(1-\lambda)\underline{R}}{1-\alpha}\right)$. $\overline{u}_2$ and $\underline{u}_2$ represent a patient depositor's utility in period 2, depending on the realization of asset returns, if there is no bank run during period 1 (i.e., $\beta = \alpha$).

The construction of the equilibrium relies on solving for a newly informed depositor's equilibrium strategies. The equilibrium strategies of an uninformed or an already informed depositor can be constructed accordingly. I will show that there exists an equilibrium in which a newly informed depositor makes his decision according to the following simple rule:

$$x_n^I = \begin{cases} 1 & \text{if impatient or } p_n^I < \widehat{p} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $n \leq N$, where $\widehat{p}$ solves

$$u_1 = \widehat{p}\overline{u}_2 + (1-\widehat{p})\,\underline{u}_2. \tag{2}$$

$\widehat{p}$ is the cutoff probability belief at which a patient depositor is indifferent between withdrawing immediately and waiting until the last period were there to be no future information about asset returns. Given the contract, the cutoff belief of a newly informed depositor is the same regardless of the stage at which the signals are received. Note that

$\widehat{p}$ is positive given $c_1 \geq 1$ and $\underline{R} \leq 1$. Also note that $\widehat{p} = 0$ if and only if $c_1 = \underline{R} = 1$ or $c_1 = \lambda = 1$.

A newly informed depositor shares the same prior with the uninformed depositors. If no one else makes a withdrawal at the current stage (which is true in equilibrium), the belief of a newly informed depositor is updated by the signal he receives:

$$p_n^I = \begin{cases} P_L(p_{n-1}^U) & \text{if } S_n = L \\ P_H(p_{n-1}^U) & \text{if } S_n = H \end{cases} \tag{3}$$

with $p_0^U = p_0$ and for $n \leq N$. If anyone else makes a withdrawal (which can happen off the equilibrium path), then $p_n^I \in [0, \underline{p})$, where $\underline{p} = P_L(\widehat{p})$.[13]

The beliefs off the equilibrium path are arbitrary. Here for convenience, a newly informed depositor's belief is assumed to be any value below $\underline{p}$ off the equilibrium path.[14] Between the end of the last stage and the beginning of the current stage, only the newly informed depositor receives new information. He would be the only one who would make a withdrawal at the beginning of the current stage. If other depositors withdraw, the newly informed depositor detects the deviation, and his belief falls below $\underline{p}$. Let $\overline{p}$ denote $P_H(\widehat{p})$. Equations $(1)-(3)$ imply that on the equilibrium path, if a patient newly informed depositor's prior is between $\underline{p}$ and $\overline{p}$, then he withdraws if he receives a low signal and waits otherwise. If his prior is above $\overline{p}$, he will not withdraw even if he gets a low signal, whereas if his prior is below $\underline{p}$, he will withdraw even if he gets a high signal.

The uninformed and already informed depositors update their beliefs by watching the action taken by the newly informed depositor. Given the newly informed depositor's strategy, the belief of an uninformed or an already informed depositor at stage n, $n \leq N$,

[13] If $\underline{p} = 0$, the off-equilibrium path belief is $p_n^I = 0$. The same rules apply to equation (4).

[14] Note that later in the proof of the proposition, an active depositors' equilibrium strategy does not rely on other depositors' off-equilibrium path beliefs. A depositor's detectable deviation from the equilibrium path (he withdraws but he is not supposed to) can trigger a bank run. However, the bank run does not affect the payoff that deviator receives because depositors are served sequentially.

is updated by

$$p_n^{\tau} = \begin{cases} [0, \underline{p}) & \text{if } X_n > 1, \text{ or } (X_n = 0 \text{ and } p_{n-1}^U < \underline{p}) \\ P_{\widetilde{L}}(p_{n-1}^{\tau}) & \text{if } X_n = 1 \text{ and } \underline{p} \leq p_{n-1}^U < \overline{p} \\ P_H(p_{n-1}^{\tau}) & \text{if } X_n = 0 \text{ and } \underline{p} \leq p_{n-1}^U < \overline{p} \\ p_{n-1}^{\tau} & \text{otherwise} \end{cases} \tag{4}$$

with $p_0^{\tau} = p_0$ and for $\tau = A, U$.

On the equilibrium path, an uninformed or an already informed depositor updates his belief by the information inferred. Off the equilibrium path, the belief is assumed to be any value below $\underline{p}$. An uninformed or an already informed depositor can detect the deviation in the following two situations: (i) more than one withdrawal is observed at the beginning of the current stage, and (ii) the newly informed depositor does not withdraw given $p_{n-1}^U < \underline{p}$. According to $(1) - (3)$, the newly informed depositor at stage n with prior $p_{n-1}^U < \underline{p}$ withdraws even if he receives a high signal (although in equilibrium, there is no active depositor with beliefs lower than $\underline{p}$). If he does not withdraw, other depositors detect the deviation.

Note that an already informed depositor's prior belief can differ from that of the newly informed and the uninformed depositors because he has received private information that others might not have perfectly inferred, whereas he observes everything others do.

At stage $N + 1$, because there is no new information about asset returns, an active depositor's belief is equal to his belief at stage N. So p_N^{τ} is a depositor's finalized belief.

With p_N^U as his finalized belief, an uninformed depositor compares his expected utilities from withdrawing and from waiting at stage N. If $p_N^U \geq \widehat{p}$, he will wait for period 2 unless he turns out to be an impatient type at stage $N + 1$. Otherwise, he will withdraw, regardless of the actions of the other depositors. If all depositors withdraw, each depositor has a chance of $1/c_1$ of getting paid, given $c_1 \geq 1$. By symmetric strategies, the expected utility of an uninformed depositor at the end of stage N is

$$V_N\left(p_N^U\right)=\begin{cases}\alpha u_1+(1-\alpha)\left[p_N^U\overline{u}_2+\left(1-p_N^U\right)\underline{u}_2\right] & \text{if } p_N^U\geq\widehat{p}\\ \dfrac{1}{c_1}u_1 & \text{otherwise.}\end{cases}\tag{5}$$

Note that due to payoff externality, which is captured by $1/c_1$ in (5), V_N is discontinuous and non-convex if $c_1>1$.

Given an uninformed depositor's expected utility at stage N and the strategies of the newly informed depositors, the expected utility of an uninformed depositor at stage n, $n<N$, can be constructed in a recursive way:

$$V_n\left(p_n^U\right)=\begin{cases}\alpha u_1+(1-\alpha)\left[p_n^U\overline{u}_2+\left(1-p_n^U\right)\underline{u}_2\right] & \text{if } p_n^U\geq\overline{p}\\ I_n\left(p_n^U\right)\max\left\{\pi\left(p_n^U\right)V_{n+1}\left(P_H\left(p_n^U\right)\right)+(1-\pi\left(p_n^U\right))V_{n+1}\left(P_{\widetilde{L}}\left(p_n^U\right)\right),u_1\right\} & \text{if } \underline{p}\leq p_n^U<\overline{p}\\ \dfrac{1}{c_1}u_1 & \text{if } p_n^U<\underline{p},\end{cases}\tag{6}$$

where

$$I_n(p)=\begin{cases}1 & \text{if } \pi(p)V_{n+1}\left(P_H(p)\right)+(1-\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right)\geq u_1\\ \dfrac{1}{c_1} & \text{otherwise}\end{cases}\tag{7}$$

captures the payoff externality when a bank run occurs, and

$$\pi(p)=(1-\alpha)\left[(1-p)(1-q)+pq\right]\tag{8}$$

is the probability that the depositor informed at the next stage receives a high signal and is also patient, given the posterior belief of p at the current stage.

In light of the foregoing expected utility, an uninformed depositor's strategy is

$$x_n^U=\begin{cases}1 & \text{if } V_n\left(p_n^U\right)<u_1\\ 0 & \text{otherwise}\end{cases}\tag{9}$$

for $n\leq N$.

If the prior at stage $n+1$ is very high (very low), that is, $p_n^U \geq \overline{p}$ ($p_n^U < \underline{p}$), then even though a low (high) signal is received, the newly informed depositor's posterior belief at stage $n+1$ is still above (below) the critical level of $\widehat{p}$. So the newly informed depositor will not withdraw[15] (wait). The newly informed depositor's action does not carry information about his signal of asset returns, so the beliefs of the uninformed depositors do not change. The same argument applies to all future stages. Because no more information will be inferred from the actions of the newly informed depositors at future stages, an uninformed depositor's belief will stay at the current level. According to his current belief, the expected utility in the last period, if he does not withdraw and bank run does not occur, is $\alpha u_1 + (1-\alpha)\left[p_n^U \overline{u}_2 + \left(1-p_n^U\right)\underline{u}_2\right]$, which is greater (lower) than u_1 as $p_n^U \geq \overline{p}$ ($p_n^U < \underline{p}$).

Suppose the newly informed depositor's prior is moderately high. If a low signal is received, the posterior belief falls below $\widehat{p}$, whereas if a high signal is received, the posterior belief is above $\widehat{p}$. When the newly informed depositor waits, his action fully reveals that he gets a high signal. The belief of the uninformed depositors will be updated by P_H accordingly. However, if a withdrawal is observed, an uninformed depositor's belief will be updated by $P_{\widetilde{L}}$ as he is not sure whether the newly informed depositor received a low signal or encountered a consumption shock. The expected utility of an uninformed depositor at the current stage is the weighted average of the possible expected utilities at the next stage, where the weights are the probabilities that his current belief will be updated by either P_H or $P_{\widetilde{L}}$ at that next stage. Whether an uninformed depositor decides to withdraw at the current stage depends on whether the weighted average exceeds u_1. When he withdraws, by the symmetric strategies and payoff externality, his expected utility is $\frac{1}{c_1}u_1$.

Note also that $V_n(p)$ is not necessarily increasing in the interval of $[\underline{p}, \overline{p})$, because it is a weighted average of the next period's possible expected utilities, $V_{n+1}(P_H(p))$ and $V_{n+1}\left(P_{\widetilde{L}}(p)\right)$, which are in the non-convex set of $V_{n+1}(p)$ by recursive construction.

An already informed patient depositor's expected utility at stage n, denoted by W_n,

[15] That is, he will not withdraw unless he is an impatient type.

can be constructed in a similar way:

$$W_N\left(p_N^A\right) = \begin{cases} \max\left\{p_N^A\overline{u}_2 + \left(1-p_N^A\right)\underline{u}_2,\ u_1\right\} & \text{if } V_N\left(p_N^U\right) \geq u_1 \\ \dfrac{1}{c_1}u_1 & \text{otherwise} \end{cases} \tag{10}$$

$$W_n\left(p_n^A\right) = \begin{cases} \max\left\{p_n^A\overline{u}_2 + \left(1-p_n^A\right)\underline{u}_2, u_1\right\} & \text{if } p_n^U \geq \overline{p} \\ \max\{\pi\left(p_n^A\right)W_{n+1}\left(P_H\left(p_n^A\right)\right) + & \text{if } \underline{p} \leq p_n^U < \overline{p} \text{ and } V_n\left(p_n^U\right) \geq u_1 \\ \quad \left(1-\pi\left(p_n^A\right)\right)W_{n+1}\left(P_{\widetilde{L}}\left(p_n^A\right)\right), u_1\} & \\ \dfrac{1}{c_1}u_1 & \text{otherwise} \end{cases} \tag{11}$$

for $n \leq N$. An already informed depositor is patient, otherwise he would have withdrawn earlier. He knows the beliefs of the uninformed depositors, and he can predict whether the uninformed depositors will withdraw or not. Because the uninformed depositors are of measure 1, when they withdraw, an already informed depositor should also do so, otherwise he will be left unpaid. Therefore, the expected utility of an already informed depositor is conditional on whether the uninformed depositors withdraw or not. The expected utility function W_n also applies to the newly informed depositor with posterior belief of p_n^I if he is a patient type.

For $n \leq N$, an already informed depositor's strategy is

$$x_n^A = \begin{cases} 1 & \text{if } W_n\left(p_n^A\right) < u_1 \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

At stage $N+1$, an active depositor's strategy is

$$x_{N+1} = \begin{cases} 1 & \text{if impatient or } p_{N+1} < \widehat{p} \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

where $p_{N+1} = p_N$.

3.2.2 Definitions and lemmas

Before proving that the conjectured equilibrium discussed above is indeed an equilibrium given a demand deposit contract, I first introduce the definitions of a herd of withdrawals and a herd of non-withdrawals and present two lemmas on the properties of an active depositor's expected utility.

Definition 1 A herd of non-withdrawals begins when (1) the newly informed depositor does not withdraw deposits unless he is impatient, even if a low signal about asset returns is received, and (2) no other depositor withdraws unless his consumption type is revealed to be impatient.

Definition 2 A herd of withdrawals begins when all depositors withdraw deposits.

The logic behind the proof of the equilibrium is similar to Chari and Kehoe (2003). However, due to payoff externality and the fact that the consumption types are private information, the following lemmas are needed to establish the properties of an active depositor's expected utility. Lemma 1 shows that uninformed depositors are willing to wait if high signals are inferred. So, in the equilibrium, a herd of withdrawals is triggered by the inference of low signals. Lemma 2 shows that if an already informed depositor and an uninformed depositor share the same belief, and the uninformed depositor is willing to wait, then the already informed depositor also is willing to wait. In the equilibrium, an already informed depositor will not run on the bank unless the uninformed depositors decide to run.

Lemma 1 Given a posterior belief of p at stage n, if $V_n(p) \geq u_1$, then $V_{n+1}(P_H(p)) \geq u_1$.[16]

By lemma 1, if a newly informed depositor's decision to wait conveys a high signal to the uninformed depositors, his decision will not trigger a bank run.

Lemma 2 If $p_n^U = p_n^A$ and $V_n(p_n^U) \geq u_1$, then $W_n(p_n^A) \geq u_1$.

[16] Proofs of lemma 1 and lemma 2 are in the appendix.

The intuition behind lemma 2 is the following. Conditional on being impatient, a depositor prefers to withdraw immediately. If an uninformed depositor is willing to wait, it must be true that conditional on being patient, the expected utility from waiting is higher than that from withdrawing immediately. An already informed depositor is patient. If he shares the same belief as the uninformed depositors, his expected utility is the same as the uninformed depositors conditional on the uninformed being patient. Therefore, the already informed depositor waits if the uninformed do so.

3.2.3 Proof of the equilibrium

Proposition Given $c_1 \geq 1$, the beliefs and strategies in $(1)-(13)$ constitute a perfect Bayesian equilibrium.[17]

Proof. By construction, an active depositor's belief is updated by Bayes' rule whenever possible. The strategies of uninformed or already informed depositors are constructed to be the equilibrium strategies given the strategies of a newly informed depositor. Hence, the proof of the equilibrium needs only show that a newly informed depositor of a patient type will follow the strategies described by $(1)-(3)$, given the strategies of the uninformed and the already informed depositors.

A newly informed depositor's prior belief at stage n is higher than $\underline{p}$. Otherwise a herd of withdrawals would have occurred already. If a herd of non-withdrawals has begun already, that is, $p_{n-1}^U \geq \overline{p}$, the newly informed depositor's actions do not change the beliefs of other depositors, and he will not be able to infer any information in future. Even if he receives a low signal, his private belief is still above $\widehat{p}$, so he will wait. In what follows, I discuss cases according to the signal that the newly informed depositor gets at stage n, given that a herd of non-withdrawals has not begun yet, that is, $\underline{p} \leq p_{n-1}^U < \overline{p}$.

[17] This equilibrium can be viewed as the limiting case of a finite economy. Consider an economy with K depositors and $N+1$ stages, where $N < K$. Suppose depositors have an alternative short-term investment opportunity, which yields a return of $1+\varepsilon$ (ε is small but positive) per stage. Let $\hat{p}_n$ be the belief of a newly informed depositor at stage n at which he is indifferent between withdrawing immediately and waiting until the last period were there to be no future information about productivity. We can list the conditions on the parameters for a perfect Bayesian equilibrium in which a newly informed depositor withdraws when his belief is below $\hat{p}_n$ or he is impatient, and waits otherwise. The strategies and beliefs of other types of depositors are constructed accordingly. When $K \to \infty$, these conditions are always satisfied. The constructed strategies and beliefs converge to $(1)-(13)$ when $K \to \infty$ and $\varepsilon \to 0$.

(1) The newly informed depositor gets a high signal. His belief now is higher than $\widehat{p}$. If he waits, he conveys the high signal to all other depositors. He becomes an already informed depositor at the next stage and shares the same belief with the uninformed depositors. By lemma 1, the uninformed depositors will be waiting. By lemma 2, the newly informed depositor will wait too.

(2) The newly informed depositor gets a low signal. His belief becomes $p_n^I = P_L\left(p_{n-1}^U\right) < \widehat{p}$. According to the strategies, he should withdraw and get c_1. Suppose he waits. Then an uninformed depositor is misled and his belief is updated to $p_n^U = P_H\left(p_{n-1}^U\right)$. The belief of the deviator now becomes two low signals below that of the uninformed depositors. That is, $p_n^I = P_L^2\left(p_n^U\right)$. (The superscript on P_L denotes the number of updates by P_L. Similar notation applies to P_H and $P_{\tilde{L}}$.) By choosing to deviate, the best outcome that the informed depositor can anticipate is a herd of non-withdrawals. (If he anticipates a herd of withdrawals to occur, he would withdraw immediately.) Suppose a herd of non-withdrawals occurs at a later stage j. The posterior belief of uninformed depositors at stage j satisfies $p_j^U \geq \overline{p}$. It also must be true that $p_{j-1}^U < \overline{p}$ or $P_L\left(p_{j-1}^U\right) < \widehat{p}$. Otherwise, the herd of non-withdrawals would have begun earlier. Since $p_{j-1}^U < p_j^U$, it must be true that a high signal is inferred at stage j. So we have $p_j^U = P_H\left(p_{j-1}^U\right)$ or $P_L\left(p_j^U\right) = p_{j-1}^U$. At stage j, the belief of the depositor who has deviated is still two low signals below that of the uninformed. That is, the deviator's belief at stage j is $P_L^2\left(p_j^U\right)$. Because $P_L^2\left(p_j^U\right) = P_L\left(p_{j-1}^U\right) < \widehat{p}$, at the stage that the herd of non-withdrawals begins, the expected utility of the deviator is still lower than u_1. Therefore, the depositor informed at stage n does not benefit from deviation. A newly informed depositor weakly prefers to withdraw immediately if a low signal about asset returns is received.

In the equilibrium, the already informed depositors who were informed before a herd of non-withdrawals begins share the same belief with the uninformed depositors. By Lemma 2, the already informed wait unless the uninformed decide to run on the bank. Those who are informed after a herd of non-withdrawals begins wait.

Because the consumption types are private information, deviations are undetectable to the uninformed and already informed depositors unless more than one withdrawal is observed at a stage before a herd of withdrawals begins. The newly informed depositor can detect deviations if anyone else makes a withdrawal at the current stage. According to the beliefs off the equilibrium path, any detected deviation triggers a bank run if $\widehat{p} > 0$. If $\widehat{p} = 0$, waiting is the dominant strategy even if all other depositors withdraw as $u_1 = \underline{u}_2 = u(1)$.[18] ■

We observe the following along the equilibrium path: A newly informed depositor withdraws if he is impatient. If he is patient, he follows his private signal about asset returns if his belief is below $\overline{p}$. Other depositors watch the actions by the newly informed depositor and update their beliefs accordingly. If there are a sufficient number of non-withdrawals, the beliefs of the uninformed depositors will be raised above $\overline{p}$, and a herd of non-withdrawals will start. In the opposite case, if many informed depositors withdraw, the beliefs of other depositors will keep falling until their expected utility reaches the threshold, u_1, at which point a herd of withdrawals starts. Although by (6) – (9) the lowest possible belief to trigger a herd of withdrawal is $\underline{p}$, a herd of withdrawals can start before the belief falls below $\underline{p}$ due to payoff externality. Section 4 discusses this aspect of the equilibrium in detail.

4 Discussion of the Equilibrium

Bank runs in this paper are partly fundamental based. Information about the fundamentals is valuable in the sense that if portfolio returns are low, early liquidation of the assets is desirable because it can avoid future losses. Because signals about the asset returns are noisy, other things being equal, a depositor wants to accumulate as much information as possible before he makes a decision. However, because signals are private, depositors can only infer the information by watching the actions of those who are informed, and the inference can only be drawn before either type of herd begins. A depositor with a higher belief, on one hand, knows that the asset returns are more likely to be high, but on the other hand, understands that the economy is more likely to reach a herd of

[18] Note that $\widehat{p} = 0$ if and only if $c^1 = \underline{R} = 1$ or $c^1 = \lambda = 1$.

non-withdrawals in which no information will be made available in the future. The former has a positive effect on expected utility, whereas the latter adds a negative effect. Consequently, a depositor's expected utility is not necessarily increasing in his belief.[19]

Because early liquidation occurs as a consequence of a bank run, it comes with a cost due to payoff externality – some depositors will not be paid if a bank run occurs. Payoff externality strengthens the positive effect of a higher belief on expected utility, because with a higher belief it is more likely that there will be a herd of non-withdrawals in which case the cost due to payoff externality will be avoided, although whether the expected utility function is monotone remains ambiguous. In what follows, I discuss in detail the welfare consequences of a herd. The focus is on the properties of the uninformed depositors' expected utility. I discuss cases according to whether the contract satisfies the "high cutoff belief condition" or the "low cutoff belief condition." The meaning of the conditions will become clear at the end of this section.

High cutoff belief condition: $\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\widehat{p})\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}(\widehat{p})\right)\underline{u}_2\right] > \dfrac{1}{c_1}u_1.$

Low cutoff belief condition: $\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\widehat{p})\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}(\widehat{p})\right)\underline{u}_2\right] \leq \dfrac{1}{c_1}u_1.$

The left-hand side of the cutoff belief conditions is an uninformed depositor's expected utility with belief $P_{\widetilde{L}}(\widehat{p})$ at stage N if no bank run occurs. The right-hand side is his expected utility when a bank run occurs. Everything else being equal, a bank run is more costly in the economy with the high cutoff belief condition because evaluated at $P_{\widetilde{L}}(\widehat{p})$, when the bank is forced to be liquidated by a run, its average payoff to a depositor is lower than what a depositor can get if it is not liquidated. In what follows, we will see that with the high (low) cutoff belief condition, the cutoff beliefs at stages before N are above (below) $\widehat{p}$.

[19] The non-monotonicity of the expected utility function in belief has been paid little attention in the literature. In the literature, herding is usually treated as a partial equilibrium problem, in which the cutoffs are determined exogenously by the assumed value of parameters. An agent's zero-one decision either perfectly reveals the signal received or both decisions carry the same amount of noise. Given an initial prior, only a few crucial probability levels (one and two signals above and below the initial prior) are needed to show the equilibrium. However, in the banking setup with a one-side signal extraction problem, the belief updated by observing a non-withdrawal is not completely offset by a withdrawal. There are 2^n number of possible posterior beliefs at stage n from ex-ante point of view. A general description of the expected utility function on the full domain of beliefs thus becomes necessary.

4.1 Case 1 – the high cutoff belief condition holds

Define a cutoff belief of $V_n(p)$ as follows:

Definition 3 We say $\widetilde{p}_n$ is a cutoff belief of $V_n(p)$ if there exist ε_1, $\varepsilon_2 > 0$ such that $V_n(p) \geq u_1$ for $p \in [\widetilde{p}_n, \widetilde{p}_n + \varepsilon_1]$, and $V_n(p) < u_1$ for $p \in [\widetilde{p}_n - \varepsilon_2, \widetilde{p}_n)$.

When the high cutoff belief condition holds, we have the following results.

Remark 1 Consider a contract that satisfies the high cutoff belief condition. Then the following are true. $V_n(p)$ is increasing in p for $1 \leq n \leq N$. There exists a unique cutoff belief, $\widetilde{p}_n$, such that $V_n(p) \geq u_1$ for $p \in [\widetilde{p}_n, 1]$, and $V_n(p) = \dfrac{1}{c_1} u_1$ for $p \in [0, \widetilde{p}_n)$. Finally, we have $\widetilde{p}_n > \hat{p}$ and $\widetilde{p}_n$ is decreasing in n for $1 \leq n < N$.[20]

Remark 1 states three facts when the high cutoff belief condition holds: (1) The expected utility of the uninformed depositors is increasing in their beliefs. Consequently, (2) there is a unique cutoff belief at each stage. (3) The cutoff belief is decreasing as time goes by.

A bank run is costly under the high cutoff belief condition. To see this, consider a p_N^U in the interval of $[P_{\widetilde{L}}(\widehat{p}), \widehat{p})$. With such a belief, a bank run takes place at stage N. The social welfare, measured by the aggregate expected utility, falls to $\frac{1}{c_1}u_1$. However, under the high cutoff belief condition, if depositors do not withdraw, the social welfare would actually be higher than that in the bank run. From the view of social welfare, the bank run is undesirable. To an individual depositor, the bank run is also undesirable because his expected payoff from early liquidation is lower than what he could get if everyone, including himself, waited. Aware of the risk of having a costly bank run at the next stage, the depositors must be more optimistic to wait for more information at stage $N-1$. Hence, the cutoff belief at stage $N-1$ is higher than $\widehat{p}$.[21] Working backward,

[20] Proofs of remarks 1 and 2 are in the appendix.

[21] Note that by equations (5) – (6) and the high cutoff belief condition, we have

$$\begin{aligned} V_{N-1}(\widehat{p}) &= \pi(\widehat{p})\{\alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\underline{u}_2]\} + (1-\pi(\widehat{p}))\frac{1}{c^1}u_1 \\ &< \pi(\widehat{p})\{\alpha u_1 + (1-\alpha)[P_H(\widehat{p})\overline{u}_2 + (1-P_H(\widehat{p}))\underline{u}_2]\} + \\ &\quad (1-\pi(\widehat{p}))\left\{\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\widehat{p})\overline{u}_2 + \left(1-P_{\widetilde{L}}(\widehat{p})\right)\underline{u}_2\right]\right\} \\ &= u_1. \end{aligned}$$

as the uncertainty of having a bank run gradually resolves, the cutoff beliefs decrease. Depositors become more and more willing to wait.

Under the high cutoff belief condition, depositors, worried about their loss in a possible future bank run, tend to withdraw early even though their beliefs are still moderately favorable. Because a bank run happens too soon, depositors never have a chance to accumulate sufficient information at any level of belief to justify that a bank run can mitigate future loss. Consequently, the negative effect of a high belief disappears, and the expected utility function becomes increasing in belief. As a result of the monotonicity, there is a unique cutoff belief at each stage above which the uninformed depositors are willing to wait and below which they will withdraw.

Example 1 An example of the expected utilities when the high cutoff belief condition holds.

The utility function and the parameters in this example are as follows: $u(c) = \frac{(c+b)^{1-\gamma}-b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$. $\overline{R} = 1.5$, $\underline{R} = 1$, $p_0 = 0.9$. $q = 0.999$. $\alpha = 0.01$. Let $c_1 = 1.04$ and $\lambda = \alpha c_1 = 0.0104$.

In this example, $V_n\left(p_n^U\right)$ is increasing in p_n^U for every stage. $\widetilde{p}_N = \widehat{p} = 0.0978$, $\widetilde{p}_n = 0.4383$ for $n = N-1$, $N-2, ..., 1$. Figure 1 shows $V_n\left(p_n^U\right)$, where $n = N$, $N-1$, $N-2$, $N-100$. In all figures in this paper, a solid line represents $V_n\left(p_n^U\right)$ and a dash line represents u_1.

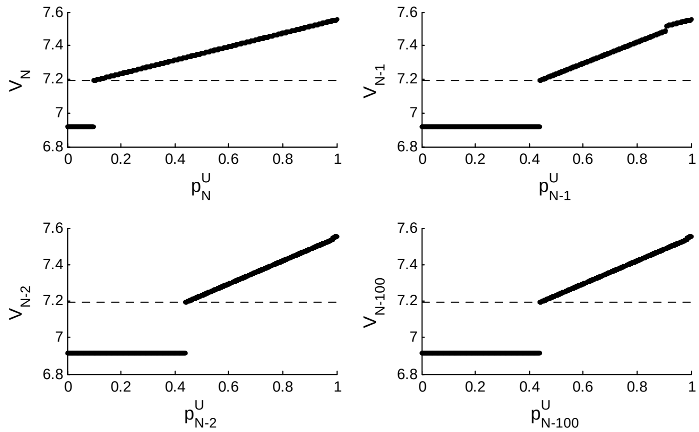

Figure 1: An example of the expected utilities when the high cutoff belief condition holds.

With the high cutoff belief condition, the sequence of $(\widetilde{p}_0, \widetilde{p}_1, ..., \widetilde{p}_{N-1}, \widehat{p}, \widehat{p})$ comprises the threshold beliefs above which the uninformed depositors wait and below which they withdraw, whereas $(\widehat{p}, \widehat{p}, ...\widehat{p}, \widehat{p}, \widehat{p})$ is the sequence of the threshold beliefs above which the newly informed depositors wait and below which they withdraw. For all depositors $(\overline{p}, \overline{p}, ..., \overline{p}, \widehat{p}, \widehat{p})$ is the sequence of beliefs above which a herd of non-withdrawals occurs at a stage.

Because $\widetilde{p}_n$ is unique and is decreasing in n, we can calculate the number of updates by $P_{\widetilde{L}}$ that are needed to trigger a bank run at stage n starting with p_0. Let a positive integer, Z_n, solve

$$P_{\widetilde{L}}^{Z_n - 1}(p_0) \geq \widetilde{p}_n, \text{ and } P_{\widetilde{L}}^{Z_n}(p_0) < \widetilde{p}_n.$$

If there have been Z_n number of withdrawals up to stage n, a bank run will take place. Because $\widetilde{p}_n \geq \widehat{p}$, a non-withdrawal triggers a herd of non-withdrawals before the beliefs of depositors fall below the cutoff.

4.2 Case 2 - the low cutoff belief condition holds

The high cutoff belief condition is a sufficient condition for a bank run to be costly. Absent such a condition, the expected utility function can exhibit non-monotonicity. We have the following results from the low cutoff belief condition.

Remark 2 Consider a contract that satisfies the low cutoff belief condition. Then the following are true: $V_n(p)$ is not necessarily increasing in p. There can be multiple cutoff beliefs. Finally, the cutoff beliefs $\widetilde{p}_n < \hat{p}$ for $1 \leq n < N$.

If the low cutoff belief condition holds, when depositors withdraw with belief of $P_{\tilde{L}}(\widehat{p})$ at stage N, the aggregate expected utility is $\frac{1}{c_1}u_1$. If they wait, however, the expected utility in the last period will be lower. Bank runs under such a circumstance serve as a valuable option. An uninformed depositor with belief $\widehat{p}$ at stage $N-1$ is willing to wait because even if a bank run occurs at the next stage (his belief would be $P_{\tilde{L}}(\widehat{p})$ then), the loss is relatively small.[22] By backward induction, the cutoff beliefs are lower than $\widehat{p}$ for any stage before N.

Example 2 An example of the expected utilities when the low cutoff belief condition holds.

The utility function and the parameters in this example are as follows: $u(c) = \frac{(c+b)^{1-\gamma}-b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$. $\overline{R} = 1.5$, $\underline{R} = 0.8$, $p_0 = 0.9$. $q = 0.9$. $\alpha = 0.01$. Let $c_1 = 1.011$, $\lambda = \alpha c_1 = 0.0101$.

Figure 2 shows $V_n(p)$, where $n = N,\ N-1,\ N-2,\ N-100$. In this example, $V_{N-100}(p)$ exhibits non-monotonicity. The cutoffs are unique at stages N, $N-1$, $N-2$, and $N-100$. $\widetilde{p}_N = \widehat{p} = 0.3716$, $\widetilde{p}_{N-1} = 0.2032$, $\widetilde{p}_{N-2} = 0.1971$, $\widetilde{p}_{N-100} = 0.1783$. However, the uniqueness of the cutoff belief is not guaranteed. We will see a case of multiple cutoff beliefs in example 3.

[22] Note that by equations (5) $-$ (6) and the low cutoff belief condition, $V_{N-1}(\widehat{p}) = \pi(\widehat{p})\{\alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\underline{u}_2]\} + (1-\pi(\widehat{p}))\frac{1}{c^1}u_1 \geq u_1$.

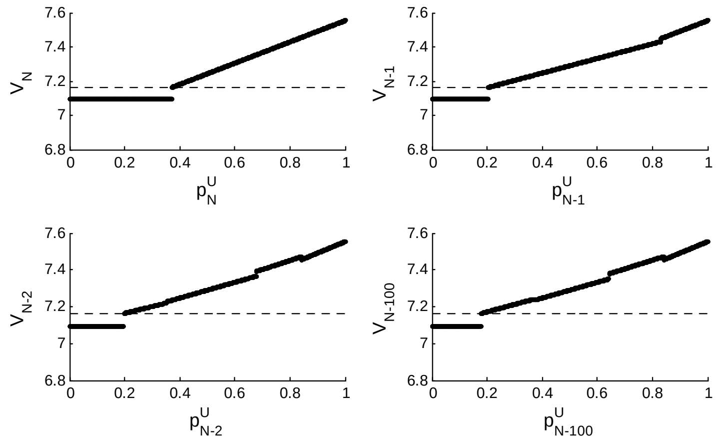

Figure 2: An example of the expected utilities when the low cutoff belief condition holds.

Multiple cutoff beliefs result from payoff externality. If there is no payoff externality (i.e., $c_1 = 1$), the option to wait guarantees a payment of 1 in period 1, regardless of other depositors' actions. Insured by such an option, an uninformed depositor prefers to wait for more information until information is no longer available. The cutoff belief is the lowest belief at which information can be revealed. (That is, if $c_1 = 1$, the cutoff beliefs at stages before stage N are $\underline{p}$.) If $c_1 > 1$, the value of the option to withdraw falls when other depositors exercise it. The uncertainty of payoff can encourage a depositor to withdraw early with a relatively higher belief because he expects there will not be much information about asset returns, but he can lose in a possible bank run. However, if he has a lower belief and he expects more information to be revealed, he could prefer to wait even though he is aware of his loss in a possible bank run. If he has an even lower belief, the dim prospect of the portfolio return and the fear of loss in a bank run can outweigh the incentive to wait.

Example 3 An example of multiple cutoff beliefs.

The utility function and the parameters in this example are as follows: $u(c) = \frac{(c+b)^{1-\gamma}-b^{1-\gamma}}{1-\gamma}$, $b = 0.01$, $\gamma = 1.5$. $\overline{R} = 2.07$, $\underline{R} = 0$, $p_0 = 0.9$. $q = 0.7$. $\alpha = 0.25$. Let

$c_1 = 1.011$ and $\lambda = \alpha c_1 = 0.2528$.

Figure 3 shows the expected utility of an uninformed depositor at the stage of $N-6$. There are two cutoffs at stage $N-6$, 0.9546 and 0.9562. If the posterior belief at stage $N-6$ falls below 0.9546 or between 0.9551 and 0.9562, the uninformed depositors will run on the bank.

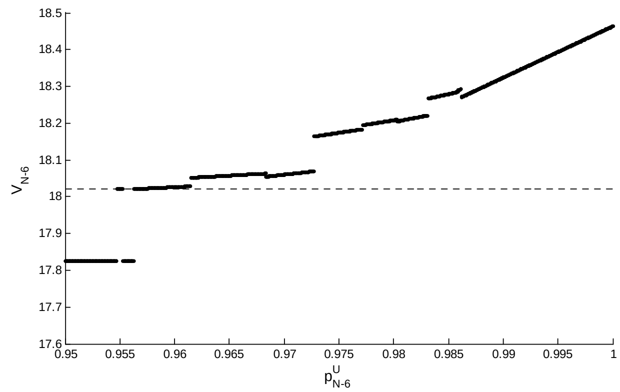

Figure 3: An example of multiple cutoff probabilities.

Multiple cutoff beliefs imply the following: Given the same contract, an economy that starts with higher initial prior p_0 can be more vulnerable to bank runs than the one with lower initial prior. A bank run may be triggered by fewer withdrawals in an economy with a higher initial probability of high asset returns than an economy with a lower one. This is because an economy with higher initial prior is more likely to reach a herd of non-withdrawals and thus has less chance to accumulate information.[23] In example 3, uninformed depositors with the belief of $p^U_{N-7} = 0.9727$ ($P_{\tilde{L}}(0.9727) = 0.9562$) run on the

[23] A question associated with multiple cutoffs is whether it is possible that a shorter queue can encourage a bank run more than a longer queue given the same parameters that describe the economy but different sequences of signals. The answer is no. To formalize the question, suppose $V_n\left(p^1\right) \geq u_1$, whereas $V_n\left(p^2\right) < u_1$. Is it possible that p^1 results from more observed withdrawals than p^2? Suppose the economy observes m withdrawals up to stage n to reach p^1, whereas it takes $m-1$ withdrawals up to stage n to reach p^2. We have $p^1 = P_L P_{\tilde{L}}\left(p^2\right)$. By remark 2, we have $p^2 < \widehat{p}$, thereby $p^1 < P_L\left(\widehat{p}\right) = \underline{p}$. By the definition of $V_n(p)$ and because $p^1 < \underline{p}$, we have $V_n\left(p^1\right) = \frac{1}{c^1} u_1$. So we reach a contradiction. Therefore, in the equilibrium, a longer queue implies that low asset return is more likely, and it encourages people to run on the bank.

bank if a withdrawal is observed at stage $N-6$, whereas if their belief is $p_{N-7}^{U}=0.9717$ ($P_{\widetilde{L}}(0.9717)=0.9547$), they prefer to wait.

Multiple cutoff beliefs have implications for empirical tests of herding effect. Runs driven by herding effect and runs driven by public information are observationally equivalent. How do we test whether a run is driven by herding? In an economy with public information about the portfolio returns, information availability does not depend on a depositor's belief. So the negative effect of having a higher belief (i.e., less chance to infer information from others' actions) that exists with herding disappears, and the likelihood of having a bank run is monotone in the strength of the fundamentals. In contrast, if the bank run is triggered by private information and herding effects, then it is possible that the likelihood of a bank run is not necessarily monotone in fundamentals. Everything else being equal, if a relatively solid bank experiences a run whereas a weak one does not, then it is due to the herding effects. However, herding in other setups without payoff externality (for example, investment herding in the traditional literature) is not testable using this method because the cutoff belief is unique in these environments, which implies that the likelihood of a herd is monotone in the fundamentals.

Without the uniqueness of the cutoffs and because a withdrawal conveys noisy information about the signal received and does not offset a non-withdrawal completely, it is difficult to describe in general the sequence of actions that triggers a herd. However, two non-withdrawals in a row will definitely trigger a herd of non-withdrawals, and two withdrawals in a row are necessary to trigger a herd of withdrawals.

4.3 Other Equilibria

The equilibrium proved in the proposition is not unique. In other equilibria, the newly informed depositors' cutoff beliefs vary with stages. For example, there can be an equilibrium in which the first informed depositor waits regardless of his private signal about the asset returns. Other newly informed depositors follow the strategies described in equation (1). Because the first informed depositor's action does not carry information, beliefs of other depositors remain unchanged. The newly informed depositor becomes an already informed depositor at the second stage and updates his expected utility according to equations $(10)-(11)$. This equilibrium exists if the first informed depositor thinks that a bank run is not likely to occur at the second stage, though his belief has been updated by a negative signal (while other depositors' beliefs stay at p_0). That is,

$$\pi\left(P_L\left(p_0\right)\right) V_2\left(P_H\left(p_0\right)\right)+\left(1-\pi\left(P_L\left(p_0\right)\right)\right) V_2\left(P_{\widetilde{L}}\left(p_0\right)\right) \geq u_1.$$

The first informed depositor will withdraw at a later stage when his expected utility falls below u_1. The other active depositors withdraw as well when they observe more than one withdrawal at a stage.

All other equilibria are constructed in the same way. In these equilibria, some of the newly informed depositors wait regardless of the signals about asset returns. Beliefs of other depositors remain unchanged at these stages. If more than one withdrawal at a stage is observed, a bank run starts.

4.4 Bank Contract

Given a contract and the equilibrium strategies, the ex-ante probability of having a herding run can be calculated by checking the probability that V_n will be lower than u_1 at each stage. The probability of having a herding run at a stage depends on the contract and other parameters. The realization of a herding run relies on the random process in which the signals are sent. Assuming depositors play the equilibrium strategies in the proposition, if a contract satisfies the high cutoff belief condition, the probability of herding runs is determined by the probability of getting Z_n number of consecutive withdrawals up to stage n. If a contract satisfies the low cutoff belief condition, it is difficult to write out the general rules of calculating the probability of herding runs. In the appendix, I calculate the probabilities of herding runs in a deposit game of $N = 2$ given that depositors play the equilibrium strategies in the proposition. A more general case can be calculated in the same way.

Because there exist multiple equilibria in the game, the choice of contract relies on the probability that a particular equilibrium occurs. The literature on bank runs employs a sunspot variable as a randomizing device in selecting which equilibrium will occur. Peck and Shell (2003) show that the ex-ante acceptable optimal contract can tolerate panic-based bank runs if the probability of having a run is small enough because the contract that admits runs also smooths consumption across depositors during normal times. Similar logic applies here. The overall probability of bank runs[24] is determined by the probability of having a run outcome in each equilibrium and by the probability that a particular equilibrium occurs. If the overall probability of runs is small, then the optimal contract can tolerate bank runs. Furthermore, a herding run admitting contract has gain and loss from imperfect information revelation: On one hand, a herding run admitting contract allows depositors to reveal their private information about the bank's portfolio performance by their actions. A weak bank thus can be liquidated early to avoid future loss. On the other hand, because the signals and the information extracted from a depositor's action are not perfect, a herding run can happen when bank's asset returns are high. In this situation, the herding run is misled.

5 Conclusion

This paper provides a model for studying detailed dynamics in bank runs. In an economy with uncertainty in asset returns, a line in front of a bank carries information about the bank's portfolio status. The formation of a line outside a bank can persuade others to join the line. In my model, a depositor makes withdrawal decisions according to his observation of the withdrawal histories of the others as well as his private information about the bank's portfolio. Given a simple demand deposit contract, there is a perfect Bayesian equilibrium in which depositors withdraw if too many withdrawals are observed and wait otherwise.

Herding runs result from imperfect private information. Given the asset return structure in this paper, it is optimal to liquidate the production early if it turns out to be

[24] Note that a panic-based run is one of the equilbiria given $c_1 > 1$, although I do not consider panic-based runs in this paper.

unsuccessful, and continue the production otherwise. The most desirable policy thus is to enforce transparency of the bank's portfolio return.

In this paper, the bank has no information advantage over the depositors, which is not quite true in reality. In a more sophisticated model in which the bank receives signals about asset returns, there arise problems such as how to eliminate the bank's moral hazard problem due to the information asymmetry between the bank and the depositors, and how the bank reduces the probability of bank runs due to the misleading signals. These can be extensions to the paper.

This paper discusses bank runs given a demand deposit contract. I do not seek a banking mechanism that eliminates herding runs. A demand deposit contract with sequential service is widely used in the banking industry.[25] It is worthwhile as a first attempt to explain the queuing process given such contracts. Green and Lin (2000, 2003) provide a model in which depositors make decisions whether to withdraw in sequence, although the depositors do not observe the line or the actions by others. They show that there exists an optimal banking contract that completely eliminates panic-based bank runs. A crucial difference between Green and Lin's economy and mine is that there is no uncertainty in asset returns in their economy. Their mechanism induces the depositors to report their consumption types truthfully by their actions. In my model, however, there are two dimensions of uncertainties: consumption uncertainty and asset return uncertainty. The withdrawal decision does not fully reveal the private information that a depositor has. Thus, there remains information asymmetry between the bank and depositors. Even if the bank provides a Green-Lin type of contract, it may not be able to eliminate herding runs.[26]

However, allowing payments to be contingent on the public withdrawal history can achieve higher social welfare (Wallace, 1988, 1990). Is there a more general banking mechanism, for example, a mechanism that induces people to report truthfully about

[25] Calomiris and Kahn (1991) show that demand-deposit contract is efficient if a bank's moral hazard problem potentially exists. Because bank runs are costly, depositors are motivated to monitor the bank and the moral hazard problem will be reduced.

[26] In a different paper (Gu, 2008), I show that in a two-depositor, two-stage economy, the Green-Lin type of mechanism does not eliminate herding runs.

their consumption types and signals about asset returns and thus reduces asymmetry in information between the bank and the depositors, that achieves a better allocation? To find a more efficient mechanism in the economy with both asset return uncertainty and consumption uncertainty is another extension of this paper, and more policy implications can be derived from the finding of such a mechanism.

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6 Appendix

6.1 Proofs of Lemmas

Lemma 1 Given a posterior belief of p at stage n, if $V_n(p) \geq u_1$, then $V_{n+1}(P_H(p)) \geq u_1$.

Proof. I will prove the lemma by discussing two cases. These two cases are called "high cutoff belief condition" and "low cutoff belief condition", respectively, in section 3.3.

Case 1: (high cutoff belief condition) $\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\widehat{p})\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}(\widehat{p})\right)\underline{u}_2\right] > \dfrac{1}{c_1}u_1$.

Claim 1: With high cutoff belief condition, we have $V_n(p) = \frac{1}{c_1}u_1$ for $p \in [0, \widehat{p})$ and $V_n(p) \leq \alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\underline{u}_2\right]$ for $p \in [\widehat{p}, 1]$.

Claim 1 is true is true for $V_n(p)$ for p on $[0, \underline{p}) \cup [\overline{p}, 1]$ by the definition of $V_n(p)$. Prove claim 1 on $[\underline{p}, \overline{p})$ by induction. It is true for $V_N(p)$ by definition. It is easy to prove for $V_{N-1}(p)$ by plugging $V_N(p)$ in $V_{N-1}(p)$ and applying the high cutoff belief condition.

Suppose claim 1 is true for every stage up to stage $n+1$.

Consider $V_n(p)$ on $[\underline{p}, \widehat{p})$. Check $\pi(p)\,V_{n+1}(P_H(p)) + (1-\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right)$.

$$
\begin{aligned}
& \pi(p)\,V_{n+1}(P_H(p)) + (1-\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right) \\
\leq\ & \pi(p)\left\{\alpha u_1 + (1-\alpha)\left[P_H(p)\,\overline{u}_2 + (1-P_H(p))\,\underline{u}_2\right]\right\} + (1-\pi(p))\frac{1}{c_1}u_1 \\
<\ & \pi(\hat{p})\left\{\alpha u_1 + (1-\alpha)\left[P_H(\hat{p})\,\overline{u}_2 + (1-P_H(\hat{p}))\,\underline{u}_2\right]\right\} + \\
& (1-\pi(\hat{p}))\left\{\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\hat{p})\,\overline{u}_2 + \left(1-P_{\widetilde{L}}(\hat{p})\right)\underline{u}_2\right]\right\} \\
=\ & u_1
\end{aligned}
$$

where the first inequality results from the fact that claim 1 is supposed to be true up to stage $n+1$, and the second inequality is by the monotonicity of $\alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\,\underline{u}_2\right]$ in p and the high cutoff belief condition. Therefore, we have $V_n(p) = \frac{1}{c_1}u_1$.

Consider $V_n(p)$ on $[\widehat{p}, \overline{p})$. If $V_n(p) = \frac{1}{c_1}u_1$, claim 1 is true by the high cutoff belief condition and the monotonicity of $\alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\,\underline{u}_2\right]$ in p. If $V_n(p) \geq u_1$ and

$P_{\widetilde{L}}(p) \in [\widehat{p}, \overline{p})$, claim 1 is true because

$$\begin{aligned}
V_n(p) &= \pi(p) V_{n+1}(P_H(p)) + (1-\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right) \\
&\leq \pi(p)\{\alpha u_1 + (1-\alpha)[P_H(p)\overline{u}_2 + (1-P_H(p))\underline{u}_2]\} + \\
&\quad (1-\pi(p))\left\{\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(p)\overline{u}_2 + \left(1-P_{\widetilde{L}}(p)\right)\underline{u}_2\right]\right\} \\
&= \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2],
\end{aligned}$$

where the inequality comes from the facts that claim 1 is supposed to be true up to stage $n+1$. Similarly, if $V_n(p) \geq u_1$ and $P_{\widetilde{L}}(p) \in [P_{\widetilde{L}}(\widehat{p}), \widehat{p})$, claim 1 is true because

$$\begin{aligned}
V_n(p) &= \pi(p) V_{n+1}(P_H(p)) + (1-\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right) \\
&\leq \pi(p)\{\alpha u_1 + (1-\alpha)[P_H(p)\overline{u}_2 + (1-P_H(p))\underline{u}_2]\} + \frac{1}{c_1}u_1 \\
&\leq \pi(p)\{\alpha u_1 + (1-\alpha)[P_H(p)\overline{u}_2 + (1-P_H(p))\underline{u}_2]\} + \\
&\quad (1-\pi(p))\left\{\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(p)\overline{u}_2 + \left(1-P_{\widetilde{L}}(p)\right)\underline{u}_2\right]\right\} \\
&= \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2],
\end{aligned}$$

where the first inequality comes from the facts that claim 1 is supposed to be true up to stage $n+1$, the second inequality results from the high cutoff belief condition and the monotonicity of $\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(p)\overline{u}_2 + \left(1-P_{\widetilde{L}}(p)\right)\underline{u}_2\right]$ in p.

By claim 1, if $V_n(p) \geq u_1$, then we must have $p \geq \widehat{p}$, thereby $P_H(p) > \overline{p}$. By the definition of V_n, $V_{n+1}(P_H(p)) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2] > u_1$.

Case 2: (low cutoff belief condition) $\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\widehat{p})\overline{u}_2 + \left(1-P_{\widetilde{L}}(\widehat{p})\right)\underline{u}_2\right] \leq \frac{1}{c_1}u_1$.

Claim 2: With low cutoff belief condition, we have $V_n(p) \geq u_1$ for $p \in [\widehat{p}, 1]$.

By definition of $V_n(p)$, claim 3 is true on $[\overline{p}, 1]$. Check $\pi(p) V_{n+1}(P_H(p)) + (1-$

$\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right)$ on $[\widehat{p},\overline{p})$.

$$\begin{aligned}
& \pi(p)\,V_{n+1}(P_H(p)) + (1-\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right) \\
\geq\ & \pi(p)\left\{\alpha u_1 + (1-\alpha)\left[P_H(p)\,\overline{u}_2 + (1-P_H(p))\,\underline{u}_2\right]\right\} + (1-\pi(p))\tfrac{1}{c_1}u_1 \\
\geq\ & \pi(\widehat{p})\left\{\alpha u_1 + (1-\alpha)\left[\overline{p}\overline{u}_2 + (1-\overline{p})\,\underline{u}_2\right]\right\} + \\
& (1-\pi(\widehat{p}))\left\{\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\widehat{p})\,\overline{u}_2 + \left(1-P_{\widetilde{L}}(\widehat{p})\right)\underline{u}_2\right]\right\} \\
=\ & \alpha u_1 + (1-\alpha)\left[\widehat{p}\overline{u}_2 + (1-\widehat{p})\,\underline{u}_2\right] = u_1
\end{aligned}$$

where the first inequality comes from the fact that $V_{n+1}\left(P_{\widetilde{L}}(p)\right)$ is bounded below by $\frac{1}{c_1}u_1$, and the second inequality results from the low cutoff belief condition and the monotonicity of $\alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\,\underline{u}_2\right]$ in p. Therefore, $V_n(p) \geq u_1$ on $[\widehat{p},\overline{p})$.

If $V_n(p) \geq u_1$, by the definition of $V_n(p)$, p must be in the interval of $\left[\underline{p},1\right]$ and $P_H(p)$ is in the interval of $[\widehat{p},1]$. By claim 2, $V_{n+1}(P_H(p)) \geq u_1$.

In both cases, we have $V_{n+1}(P_H(p)) \geq u_1$, if $V_n(p) \geq u_1$. ■

Lemma 2 If $p_n^U = p_n^A$ and $V_n\left(p_n^U\right) \geq u_1$, then $W_n\left(p_n^A\right) \geq u_1$.

Proof. Let $p_n^U = p_n^A = p$. I will prove the following claim.

Claim 3: If $V_n(p) \geq u_1$, then $V_n(p)$ can be written as

$$V_n(p) = \alpha\left[\rho_n(p)\,u_1 + (1-\rho_n(p))\tfrac{1}{c_1}u_1\right] + (1-\alpha)\,W_n(p),$$

where $\rho_n(p) \in [0,1]$.

It is obvious that claim 3 implies lemma 2.

Prove claim 3 by induction. Begin with stage N. It is easy to see that if $V_N(p) \geq u_1$, we can write $V_N(p) = \alpha u_1 + (1-\alpha)\,W_N(p)$, where $W_N(p) = p\overline{u}_2 + (1-p)\,\underline{u}_2 \geq u_1$ and $\rho_N = 1$.

Suppose it is true for every stage up to stage $n+1$.

Prove for stage n. Let $V_n(p) \geq u_1$. If $p \geq \overline{p}$, it is easy to see that $V_n(p) \geq u_1$ and $\rho_n(p) = 1$. If $p < \overline{p}$, then $V_n(p) = \pi(p)\,V_{n+1}(P_H(p)) + (1-\pi(p))V_{n+1}\left(P_{\widetilde{L}}(p)\right)$. By lemma

1, we have $V_{n+1}(P_H(p)) \geq u_1$. Suppose $V_{n+1}\left(P_{\widetilde{L}}(p)\right) \geq u_1$. By the fact that claim 3 is supposed to be true at stage $n+1$, we have $W_{n+1}(P_H(p)) \geq u_1$ and $W_{n+1}\left(P_{\widetilde{L}}(p)\right) \geq u_1$. So $W_n(p) = \pi(p) W_{n+1}(P_H(p)) + (1-\pi(p)) W_{n+1}\left(P_{\widetilde{L}}(p)\right) \geq u_1$, and

$$\begin{aligned}
V_n(p) &= \pi(p) V_{n+1}(P_H(p)) + (1-\pi(p)) V_{n+1}\left(P_{\widetilde{L}}(p)\right) \\
&= \pi(p)\left\{\alpha\left[\rho_{n+1}(P_H(p)) u_1 + (1-\rho_{n+1}(P_H(p)))\tfrac{1}{c_1}u_1\right] + (1-\alpha) W_{n+1}(P_H(p))\right\} + \\
&\quad +(1-\pi(p))\left\{\alpha\left[\rho_{n+1}\left(P_{\widetilde{L}}(p)\right) u_1 + \left(1-\rho_{n+1}\left(P_{\widetilde{L}}(p)\right)\right)\tfrac{1}{c_1}u_1\right] + (1-\alpha) W_{n+1}\left(P_{\widetilde{L}}(p)\right)\right\} \\
&= \alpha\left[\rho_n(p) u_1 + (1-\rho_n(p))\tfrac{1}{c_1}u_1\right] + (1-\alpha) W_{n+1}(p),
\end{aligned}$$

where $\rho_n = \pi(p)\rho_{n+1}(P_H(p)) + (1-\pi(p))\rho_{n+1}\left(P_{\widetilde{L}}(p)\right) \leq 1$.

Suppose alternatively that $V_{n+1}\left(P_{\widetilde{L}}(p)\right) = \frac{1}{c_1}u_1$, then we have $W_{n+1}\left(P_{\widetilde{L}}(p)\right) = \frac{1}{c_1}u_1$ by the definition of W_{n+1}. We can write $V_n(p)$ as

$$\begin{aligned}
V_n(p) &= \pi(p) V_{n+1}(P_H(p)) + (1-\pi(p)) V_{n+1}\left(P_{\widetilde{L}}(p)\right) \\
&= \pi(p) V_{n+1}(P_H(p)) + (1-\pi(p))\tfrac{1}{c_1}u_1 \\
&= \pi(p)\left\{\begin{array}{c}\alpha\left[\rho_{n+1}(p_H(p)) u_1 + (1-\rho_{n+1}(p_H(p)))\tfrac{1}{c_1}u_1\right] + \\ (1-\alpha) V_{n+1}(P_H(p))\end{array}\right\} + (1-\pi(p))\tfrac{1}{c_1}u_1 \\
&= \alpha\left[\rho_n(p) u_1 + (1-\rho_n(p))\tfrac{1}{c_1}u_1\right] + (1-\alpha)\left[\pi(p) W_{n+1}(P_H(p)) + (1-\pi(p))\tfrac{1}{c_1}u_1\right] \\
&= \alpha\left[\rho_n(p) u_1 + (1-\rho_n(p))\tfrac{1}{c_1}u_1\right] + (1-\alpha) V_n(p),
\end{aligned}$$

where $\rho_n(p) = \pi(p)\rho_{n+1}(P_H(p)) \leq 1$. ■

Remark 1 Consider a contract that satisfies the high cutoff belief condition. Then the following are true. $V_n(p)$ is increasing in p for $1 \leq n \leq N$. There exists a unique cutoff belief, $\widetilde{p}_n$, such that $V_n(p) \geq u_1$ for $p \in [\widetilde{p}_n, 1]$, and $V_n(p) = \dfrac{1}{c_1}u_1$ for $p \in [0, \widetilde{p}_n)$. Finally, we have $\widetilde{p}_n > \hat{p}$ and $\widetilde{p}_n$ is decreasing in n for $1 \leq n < N$.

Proof. By the proof of Claim 1 in Lemma 1, we have established that $V_n(p) \leq \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for p on $[\widehat{p}, 1]$ and that $\widetilde{p}_n > \widehat{p}$. In what follows, I will prove the monotonicity of $V_n(p)$ in p and the monotonicity of $\widetilde{p}_n$ in n.

(1) $V_n(p)$ is increasing in p on $[0, 1]$.

By definition, $V_n(p) = \frac{1}{c_1}u_1$ for $p \in [0, \underline{p}]$, where $\frac{1}{c_1}u_1$ is the lowest bound on $V_n(p)$. Only need to check the monotonicity for p on $[\underline{p}, 1]$. Also note that $V_n(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ is increasing in p for $p \in [\overline{p}, 1]$. Hence, we only need to prove the monotonicity of $V_n(p)$ on the interval of $[\underline{p}, \overline{p})$ and at $\overline{p}$.

Prove by induction. By checking its definition, we see that $V_N(p)$ is increasing in p. It is easy to see that $V_{N-1}(p)$ is increasing in p on $[\underline{p}, \overline{p})$ by plugging $V_N(p)$ into $V_{N-1}(p)$ and applying the monotonicity of $V_N(p)$. Because $\lim_{p \to \overline{p}^-} V_{N-1}(p) = \lim_{p \to \overline{p}^-} \left[\pi(p) V_N(P_H(p)) + (1-\pi(p)) V_N\left(P_{\widetilde{L}}(p)\right)\right] = \alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\underline{u}_2]$, $V_{N-1}(p)$ is increasing on the entire domain of $[0,1]$.

Suppose it is true for every stage up to stage $n+1$.

Check $V_n(p)$. For p on $[\underline{p}, \overline{p})$, by the monotonicity of $V_{n+1}(p)$, $\pi(p) V_{n+1}(P_H(p)) + (1-\pi(p)) V_{n+1}\left(P_{\widetilde{L}}(p)\right)$ is increasing in p. Because $V_n(p) \leq \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for $p \in [\widehat{p}, 1]$ by claim 1 in lemma 1, we have

$$\lim_{p \to \overline{p}^-} V_n(p) = \pi(\overline{p}) V_{n+1}(P_H(\overline{p})) + (1-\pi(\overline{p})) V_{n+1}\left(P_{\widetilde{L}}(\overline{p})\right) \leq \alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\underline{u}_2].$$

Therefore, $V_n(p)$ is increasing on the entire interval of $[0,1]$.

Because $V_n(p)$ is increasing on $[0,1]$, a unique cutoff probability $\widetilde{p}_n$ exists.

(2) $\widetilde{p}_n$ is decreasing in n.

Let $P_{\widetilde{H}}(p)$ be the inverse function of $P_{\widetilde{L}}(p)$. It is easy to prove that $V_n(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for $p \geq \min\left\{P_{\widetilde{H}}^{N-n}(\widehat{p}), \overline{p}\right\}$ by applying the definition of $V_n(p)$ recursively. Therefore, we have $\widetilde{p}_n \leq \min\left\{P_{\widetilde{H}}^{N-n}(\widehat{p}), \overline{p}\right\}$.

Prove the monotonicity of $\widetilde{p}_n$ by induction. By definition, $\widetilde{p}_N = \widehat{p}$. By the high cutoff probability condition, $\widehat{p} < \widetilde{p}_{N-1}$. Hence, we have $\widehat{p} < \widetilde{p}_{N-1} \leq P_{\widetilde{H}}(\widehat{p}) < \overline{p}$.

Suppose it is true for every stage up to stage $n+1$ that $\widehat{p} < \widetilde{p}_{n+2} \leq \widetilde{p}_{n+1} \leq \min\left\{P_{\widetilde{H}}^{N-(n+1)}(\widehat{p}), \overline{p}\right\}$.

Check stage n by plugging $\widetilde{p}_{n+1}$ into $V_n(p)$. Check

$$\pi(\widetilde{p}_{n+1}) V_{n+1}(P_H(\widetilde{p}_{n+1})) + (1-\pi(\widetilde{p}_{n+1})) V_{n+1}\left(P_{\widetilde{L}}(\widetilde{p}_{n+1})\right)$$
$$= \pi(\widetilde{p}_{n+1}) \{\alpha u_1 + (1-\alpha)[P_H(\widetilde{p}_{n+1}) \overline{u}_2 + (1-P_H(\widetilde{p}_{n+1})) \underline{u}_2]\} + (1-\pi(\widetilde{p}_{n+1})) \tfrac{1}{c_1} u_1. \tag{a1}$$

(a) If $V_{n+1}(\widetilde{p}_{n+1}) = u_1$, then $V_n(\widetilde{p}_{n+1}) = u_1$ because $\widetilde{p}_{n+1}$ solves the same problem that (a1) $= u_1$. Hence, we have $\widetilde{p}_n = \widetilde{p}_{n+1} \leq \min\left\{P_{\widetilde{H}}^{N-(n+1)}(\widehat{p}), \overline{p}\right\} \leq \min\left\{P_{\widetilde{H}}^{N-n}(\widehat{p}), \overline{p}\right\}$.

(b) If $V_{n+1}(\widetilde{p}_{n+1}) > u_1$, then it must be true that $\widetilde{p}_{n+1} = \min\left\{P_{\widetilde{H}}^{N-(n+1)}(\widehat{p}), \overline{p}\right\}$ and (a1) $< u_1$.Otherwise, we could have found a cutoff that is less than $\widetilde{p}_{n+1}$ for stage $n+1$. Therefore, we have $V_n(\widetilde{p}_{n+1}) < u_1$, and $\widetilde{p}_n > \widetilde{p}_{n+1}$ by the monotonicity of $V_n(p)$. ■

Remark 2 Consider a contract that satisfies the low cutoff belief condition. Then the following are true: $V_n(p)$ is not necessarily increasing in p. There can be multiple cutoff beliefs. Finally, the cutoff beliefs $\widetilde{p}_n < \hat{p}$ for $1 \leq n < N$.

Proof. The non-monotonicity of $V_n(p)$ and the multiple cutoff beliefs are proved by examples (see examples 2 and 3). By claim 2 in the proof of lemma 1, we have $\widetilde{p}_n < \hat{p}$ for $1 \leq n < N$. ■

7 Bank contract

In this section, I calculate examples of an optimal demand deposit contract, assuming depositors only play the equilibrium strategies in the proposition. Note that given a demand deposit contract, there are multiple equilibria in period 1. The literature uses a sunspot variable as an equilibrium selection device. Here, I simply assume that only the equilibrium described by (1) – (13) occurs in period 1. In this regard, the term "optimal contract" has limited meaning.

In the static bank-runs model, a feasible contract should satisfy the participation incentive compatibility constraint, which says that given all other patient depositors do not withdraw the deposits, an individual patient depositor prefers to wait. In the dynamic setup, a bank run can occur at any stage, but a feasible contract should at least give depositors the incentive to wait before anyone gets a signal. The participation incentive compatibility constraint is

$$V_0\left(p_0\right) \geq u_1. \tag{14}$$

The participation incentive compatibility constraint in the traditional Diamond-Dybvig model is a special case here, with $N = 0$ and $p_0 = 1$.

The bank chooses a contract to offer. There are two classes of contracts available to the bank: herding-run-proof contracts and herding-run-admitting contracts. A herding-run-proof contract guarantees that whichever signals are sent in period 1, the expected utility of the uninformed depositors never falls below the threshold at any stage.

7.1 Herding-run-proof contracts

A herding-run-proof contract is in any one of the three cases in my model:

Case 1: A contract that provides $c_1 < 1$. All patient depositors wait until stage $N+1$ to make decisions according to their beliefs and consumption types. No information can be inferred from the action of a newly informed depositor. The belief of an uninformed depositor is p_0 at all stages. The expected utility of an uninformed depositor at each stage is

$$V_n\left(p_0\right) = \alpha u_1 + (1-\alpha)\left[p_0 \overline{u}_2 + \left(1-p_0\right) \underline{u}_2\right]$$

for $0 \leq n \leq N$.

Case 2: $c_1 \geq 1$, and

$$P_L\left(p_0\right) \overline{u}_2 + \left(1 - P_L\left(p_0\right)\right) \underline{u}_2 \geq u_1. \tag{15}$$

That is, the initial belief is already above $\overline{p}$. A herd of non-withdrawals has already begun before anyone gets signals. The uninformed depositors never update their beliefs by the observed actions.

Case 3: $c_1 \geq 1$, and

$$P_L(p_0)\,\overline{u}_2 + (1 - P_L(p_0))\,\underline{u}_2 < u_1, \tag{16}$$

$$V_n\left(P_{\widetilde{L}}^n(p_0)\right) \geq u_1 \;\forall 0 \leq n \leq N. \tag{17}$$

That is, a newly informed depositor withdraws if a low signal is received. However, because there are too few stages and/or because the probability of being impatient is high, even though the beliefs are updated by $P_{\widetilde{L}}$ at every stage, the beliefs of the uninformed depositors are still above the thresholds. Note that (17) implies that $V_n\left(p_n^U\right) = \alpha u_1 + (1-\alpha)\left[p_n^U \overline{u}_2 + \left(1 - p_n^U\right)\underline{u}_2\right]$ for $0 \leq n \leq N$ and for any p_n^U derived from p_0, and that (17) can be rewritten as

$$P_{\widetilde{L}}^N(p_0) \geq \widehat{p}. \tag{17'}$$

The best herding-run-proof contract solves

$$\begin{aligned}
&\max\nolimits_{c_1,\lambda} V_0(p_0) = \alpha u_1 + (1-\alpha)\left[p_0 \overline{u}_2 + (1-p_0)\,\underline{u}_2\right] \\
&s.t. \quad c_1 < 1, \qquad\qquad\qquad\qquad\qquad \text{or} \\
&\qquad\quad c_1 \geq 1 \text{ and } (14) - (15), \qquad\quad \text{or} \\
&\qquad\quad c_1 \geq 1,\ (14),\ \text{and } (16) - (17).
\end{aligned}$$

7.2 Herding-run-admitting contracts

A herding-run-admitting contract admits a herd of withdrawals because $V_n\left(p_n^U\right) < u_1$ at at least one stage for some realization of p_n^U derived from p_0. The *ex-ante* probability of having a herding run given a contract can be calculated by checking the probability that $V_n\left(p_n^U\right)$ will be lower than u_1 at each stage. The probability of having a herding run at a stage depends on the contract and other parameters. The realization of a herding run relies on the random process in which the signals are sent. If a contract satisfies the high cutoff belief condition, the probability of herding runs is determined by the probability of getting Z_n number of consecutive withdrawals up to stage n. If a contract satisfies the low cutoff belief condition, it is difficult to write out the general rules of calculating the probability of herding runs. In this section, a game of $N = 2$ is calculated. A more general case can be calculated in the same way. There are five cases for a herding-run-admitting contract for $N = 2$, depending on the conditions with which a herd of withdrawals starts.

Case I: $P_{\tilde{L}}(p_0) \geq \tilde{p}_1$, $P_H(p_0) \geq \bar{p}$, $P_{\tilde{L}}^2(p_0) < \hat{p}$, $P_H P_{\tilde{L}}(p_0) \geq \hat{p}$: A herd of non-withdrawals begins if the first informed depositor waits. If both the first and the second informed depositors withdraw, then a run occurs. If the first withdraws and the second waits, a herd of non-withdrawals begins. The probability of bank runs is

$$\sigma_1 = (1 - \pi(p_0)) \left(1 - \pi\left(P_{\tilde{L}}(p_0)\right)\right).$$

Case II: $P_{\tilde{L}}(p_0) \geq \tilde{p}_1$, $P_H(p_0) < \bar{p}$, $P_{\tilde{L}}^2(p_0) < \hat{p}$, $P_H P_{\tilde{L}}(p_0) \geq \hat{p}$: No herd occurs at the first stage. If both the first and the second informed depositors withdraw, a herd of withdrawals occurs. If the first informed depositor waits, the second depositor follows his private signals, but the uninformed depositors do not withdraw regardless of the second depositor's action. The probability of bank runs is σ_1.

Case III: $P_{\tilde{L}}(p_0) < \tilde{p}_1$, $P_H(p_0) \geq \bar{p}$: A herd of withdrawals begins if the first informed depositor withdraws. A herd of non-withdrawals begins if the first informed depositor waits. The probability of herding runs is

$$\sigma_2 = 1 - \pi(p_0).$$

Case IV: $P_{\tilde{L}}(p_0) < \tilde{p}_1$, $P_H(p_0) < \bar{p}$, $P_H P_{\tilde{L}}(p_0) \geq \hat{p}$: A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor follows his private signals. However, the uninformed depositors do not withdraw regardless of the second depositor's decision. The probability of herding runs is σ_2.

Case V: $P_{\tilde{L}}(p_0) < \tilde{p}_1$, $P_H(p_0) < \bar{p}$, $P_H P_{\tilde{L}}(p_0) < \hat{p}$: A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor still follows his private signals. The uninformed depositors wait if the second depositor waits, and they withdraw if the second depositor withdraws. The probability of herding runs is

$$\sigma_3 = 1 - \pi(p_0) + \pi(p_0)(1 - \pi(P_H(p_0))).$$

7.2.1 Conditions for herding-run-admitting contracts $(N = 2)$

The conditions for the five cases of herding-run-admitting contracts are listed in this section. A herding-run-admitting contract should at least satisfy (14) and the following:

$$\begin{aligned} P_{\widetilde{L}}^{2}(p_0)\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}^{2}(p_0)\right)\underline{u}_2 &\leq u_1, \text{ and} \quad (18)\\ P_{H}^{2}(p_0)\,\overline{u}_2 + \left(1 - P_{H}^{2}(p_0)\right)\underline{u}_2 &> u_1, \quad (19) \end{aligned}$$

which imply $V_2\left(P_{\widetilde{L}}^{2}(p_0)\right) \leq u_1$ and $V_2\left(P_{H}^{2}(p_0)\right) > u_1$, respectively.

The feasible contract also implies $V_1\left(P_H(p_0)\right) > u_1$ by lemma 1. I first list the conditions for all of the possible outcomes after each newly informed depositor's decision is observed.

1. If the first informed depositor waits, a herd of non-withdrawals occurs.

$$P_L P_H(p_0)\,\overline{u}_2 + \left(1 - P_L P_H(p_0)\right)\underline{u}_2 = p_0\overline{u}_2 + (1 - p_0)\,\underline{u}_2 \geq u_1. \quad (20)$$

2. If the first informed depositor withdraws, a herd of withdrawals occurs.

$$V_1\left(P_{\widetilde{L}}(p_0)\right) < u_1.$$

3. If the first informed depositor withdraws, a herd of withdrawals does not occur. The second depositor follows the signal as $P_L P_{\widetilde{L}}(p_0)\,\overline{u}_2 + \left(1 - P_L P_{\widetilde{L}}(p_0)\right)\underline{u}_2 < u_1$, guaranteed by (18). The uninformed depositors withdraw if the second depositor withdraws (by (18)), and they wait if the second depositor waits.

$$\begin{aligned} V_1\left(P_{\widetilde{L}}(p_0)\right) &\geq u_1\\ V_2\left(P_H P_{\widetilde{L}}(p_0)\right) &= \alpha u_1 + (1-\alpha)\left[P_H P_{\widetilde{L}}(p_0)\,\overline{u}_2 + \left(1 - P_H P_{\widetilde{L}}(p_0)\right)\underline{u}_2\right] \geq u_1 \quad (21) \end{aligned}$$

4. If the first informed depositor waits, a herd of non-withdrawals does not occur. The second depositor follows the signal. The uninformed depositors withdraw if the

second depositor withdraws, and they wait if the second depositor waits.

$$\begin{aligned} p_0 \overline{u}_2 + (1-p_0)\, \underline{u}_2 &< u_1, \text{ and} \quad (22) \\ \alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}} P_H(p_0)\, \overline{u}_2 + \left(1 - P_{\widetilde{L}} P_H(p_0)\right) \underline{u}_2\right] &< u_1. \quad (23) \end{aligned}$$

5. If the first informed depositor waits, a herd of non-withdrawals does not occur. The second depositor follows the signal. The uninformed depositors wait regardless of the second depositor's decision, that is, $(21) - (22)$.

The combinations of the above five outcomes constitute descriptions of equilibrium outcomes given the contract.

Case I: Combine 1 and 3.

The probability of bank runs is $\sigma_1 = (1 - \pi(p_0))\left(1 - \pi\left(P_{\widetilde{L}}(p_0)\right)\right)$.

Equations $(18) - (20)$ are necessarily required for the outcome. The participation incentive constraint is

$$V_0(p_0) = \pi(p_0)\, V_1(P_H(p_0)) + (1 - \pi(p_0))\, V_1\left(P_{\widetilde{L}}(p_0)\right) \geq u_1 \quad (24)$$

where

$$\begin{aligned} V_1\left(P_{\widetilde{L}}(p_0)\right) &= \pi\left(P_{\widetilde{L}}(p_0)\right)\left\{\alpha u_1 + (1-\alpha)\left[P_H P_{\widetilde{L}}(p_0)\, \overline{u}_2 + \left(1 - P_H P_{\widetilde{L}}(p_0)\right) \underline{u}_2\right]\right\} + \\ &\quad \left(1 - \pi\left(P_{\widetilde{L}}(p_0)\right)\right) \tfrac{1}{c_1} u_1 \quad (25) \\ &\geq u_1, \end{aligned}$$

and

$$V_1(P_H(p_0)) = \alpha u_1 + (1-\alpha)\left[P_H(p_0)\, \overline{u}_2 + (1 - P_H(p_0))\, \underline{u}_2\right] \geq u_1. \quad (26)$$

(26) is guaranteed by (20).

The *ex-ante* expected utility maximization problem is

$$\max_{c_1,\lambda} V_0(p_0)$$
$$s.t.\ c_1 \geq 1,\ \ (18)-(20),(24)-(26).$$

Case II: Combine 3 and 5.

The probability of bank runs is σ_1. The conditions for the outcome are $(18)-(19),(21)-(22)$, and $(24)-(26)$, where (26) is guaranteed by (21) in this case. The *ex-ante* expected utility maximization problem is

$$\max_{c_1,\lambda} V_0(p_0)$$
$$s.t.\ c_1 \geq 1,\ \ (18)-(19),\ \ (21)-(22),\ \text{and}\ \ (24)-(26).$$

Case III: Combine 1 and 2.

The probability of bank runs is $\sigma_2 = 1-\pi(p_0)$. The conditions for the outcome are $(18)-(20)$, and (24). In addition, the participation incentive constraint requires

$$V_0(p_0) = \pi(p_0) V_1(P_H(p_0)) + (1-\pi(p_0)) V_1\left(P_{\widetilde{L}}(p_0)\right) \geq u_1,$$

where

$$V_1(P_H(p_0)) = \alpha u_1 + (1-\alpha)\left[P_H(p_0)\overline{u}_2 + (1-P_H(p_0))\underline{u}_2\right] \geq u_1 \tag{27}$$

is guaranteed by (20), and $V_1\left(P_{\widetilde{L}}(p_0)\right) = \frac{1}{c_1}u_1$ implies

$$\pi\left(P_{\widetilde{L}}(p_0)\right)\left\{\alpha u_1 + (1-\alpha)\left[P_H P_{\widetilde{L}}(p_0)\overline{u}_2 + \left(1-P_H P_{\widetilde{L}}(p_0)\right)\underline{u}_2\right]\right\} + \left(1-\pi\left(P_{\widetilde{L}}(p_0)\right)\right)\frac{1}{c_1}u_1 < u_1. \tag{28}$$

The *ex-ante* expected utility maximization problem is

$$\max_{c_1,\lambda} V_0(p_0)$$
$$s.t. c_1 \geq 1,\ \ (18)-(20),\ \ (24),\ \text{and}\ \ (27)-(28).$$

Case IV: Combine 2 and 5.

The probability of bank runs is σ_2. The conditions for the outcome are $(18)-(19)$, $(21)-(22)$, (24) and $(27)-(28)$, where (27) is guaranteed by (21). The *ex-ante* expected utility maximization problem is

$$\begin{aligned}&\max_{c_1,\lambda} V_0\left(p_0\right)\\&s.t. c_1 \geq 1,\ \ (18)-(19),\ \ (21)-(22),\ \ (24),\ \ (27)-(28).\end{aligned}$$

Case V: Combine 2 and 4.

The probability of bank runs is $\sigma_3 = 1-\pi\left(p_0\right)+\pi\left(p_0\right)\left(1-\pi\left(P_H\left(p_0\right)\right)\right)$. The conditions for the outcome are (14), $(18)-(19)$, and $(22)-(24)$. The participation incentive constraint requires:

$$V_0\left(p_0\right)=\pi\left(p_0\right)V_1\left(P_H\left(p_0\right)\right)+\left(1-\pi\left(p_0\right)\right)V_1\left(P_{\widetilde{L}}\left(p_0\right)\right)\geq u_1$$

where

$$\begin{aligned}V_1\left(P_H\left(p_0\right)\right) &= \pi\left(P_H\left(p_0\right)\right)\left\{\alpha v_1+(1-\alpha)\left[P_H^2\left(p_0\right)\overline{u}_2+\left(1-P_H^2\left(p_0\right)\right)\underline{u}_2\right]\right\}+ \\ &\quad \left(1-\pi\left(P_H\left(p_0\right)\right)\right)\tfrac{1}{c_1}u_1 \\ &\geq u_1,\end{aligned} \tag{29}$$

and $V_1\left(P_{\widetilde{L}}\left(p_0\right)\right)=\frac{1}{c_1}u_1$, guaranteed by (23).

The *ex-ante* expected utility maximization problem is

$$\begin{aligned}&\max_{c_1,\lambda} V_0\left(p_0\right)\\&s.t. c_1 \geq 1,\ (18)-(19),(22)-(24),(29).\end{aligned}$$

A competitive bank chooses the optimal contract from the classes of herding-run-proof and herding-run-admitting contracts. There are three factors concerning which type of contract to offer. First, because a herding-run-proof contract is associated with lower

c_1, whereas a herding-run-admitting contract provides higher c_1, a herding-run-admitting contract helps smooth consumptions across consumption types. This is a positive side of providing a herding-run-admitting contract. Second, a herding-run-admitting contract allows depositors to reveal their private information about the bank's portfolio performance by their actions. It is again a positive side of a herding-run-admitting contract. Third, because the signals and the information extracted from a depositor's action are not perfect, a bank run can happen when bank's asset returns are high. This is a negative side of a herding-run-admitting contract. Which contract to provide depends on the overall effects of the three.

The choice among herding-run-admitting contracts also depends on several factors. First, a higher c_1 helps smooth consumptions, but it is usually associated with higher probability of herding runs and lower social welfare in bank runs. The second factor is unique to a sequential-move game. The optimal herding-run-admitting contract should allow as much information as possible to be sensed publicly before any type of herd begins. The first N depositors can be treated as experiments. The result of each experiment can only be read before herds begin. A careful choice of contract should prolong the effective experiment process as much as possible. High c_1 and low c_2 can encourage people to run on the bank, and a bank run can happen too soon.

I compute two examples to illustrate that in some economies a run-admitting contract is optimal, whereas in other economies a herding-run-proof contract is optimal. I compute the best contract in each of the three herding-run-proof cases and the five herding-run-admitting cases. The optimal contract is the best of the best.

In an economy without information about asset returns, the bank chooses a contract to maximize $\alpha u_1 + (1-\alpha)\left[p_0\overline{u}_2 + (1-p_0)\underline{u}_2\right]$, subject to the incentive compatibility constraint $p_0\overline{u}_2 + (1-p_0)\underline{u}_2 \geq u_1$. If herding runs are undesirable under the optimal demand deposit contract, the bank may want to use a "curtain" to prevent depositors from seeing each others' actions. From the examples below, we will see that information inferred from others' actions can improve *ex-ante* welfare.

An individual depositor's expected utility in autarky is $u(1)$. If the optimal banking contract is accepted *ex ante*, $V_0(p_0)$ must be at least equal to $u(1)$.

7.3 Computed Examples

Parameters and functions used in examples 4 and 5 are $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$. $\overline{R} = 1.5$, $\underline{R} = 0.2$, $p_0 = 0.99$. $q = 0.99$.

Example 4 $\alpha = 0.01$.

The optimal contract in each case is listed in Table 1. A herding-run-admitting contract is optimal in this example, mainly because it induces depositors to reveal the signals they receive. This is also the reason the economy that allows for herding runs can achieve higher *ex-ante* welfare than the economy with no information about asset returns.

Table 1: An example of an optimal contract that permits herding runs

	σ	c_1	λ	$V_0(p_0)$
Autarky	0	1.0000	1	7.1529
Banking economy with no info	0	1.0001	0.0100	7.5332
Best herding-run-proof contract in case 1	0	1.0000	0.0100	7.5332
Best herding-run-proof contract in case 2	0	1.0000	1.0000	7.1529
Best herding-run-proof contract in case 3	0	1.0000	1.0000	7.1529
Best herding-run-admitting contract in case I	0.0102	1.0000	0.0100	7.5487*
Best herding-run-admitting contract in case II	0.0102	1.0000	1.0000	7.1529
Best herding-run-admitting contract in case III	0.0296	1.0876	0.0109	7.5263
Best herding-run-admitting contract in case IV	0.0296	1.0000	1.0000	7.1529
Best herding-run-admitting contract in case V	0.0490	1.4868	0.0149	7.4310

Example 5 $\alpha = 0.2$.

The optimal contract in each case is listed in Table 2. In this example, a herding-run-proof contract is optimal. Under a herding-run-admitting contract, the probability of bank runs increases in α because the probability of observing withdrawals is raised. An informed depositor's withdrawal action carries noisier information about the signals he receives, and bank runs happen too often when the asset returns are high. In addition, because there are more impatient depositors in the economy, the payments to depositors in period 1 decrease due to the resource constraint, which leaves more room for using a herding-run-proof contract.

Table 2: An example of an optimal contract that is run-proof

	σ	c_1	λ	$V_0(p_0)$
Autarky	0	1	1	7.1529
Banking economy with no info	0	1.0028	0.2006	7.4602
Best herding-run-proof contract in case 1	0	1.0000	0.2000	7.4602*
Best herding-run-proof contract in case 2	0	1.0000	1.0000	7.1529
Best herding-run-proof contract in case 3	0	1.0028	0.2006	7.4602*
Best herding-run-admitting contract in case I	0.0527	1.0213	0.2043	7.4523
Best herding-run-admitting contract in case II	0.0527	1.0000	1.0000	7.1529
Best herding-run-admitting contract in case III	0.2158	1.1047	0.2209	7.2785
Best herding-run-admitting contract in case IV	0.2158	1.0000	1.0000	7.1529
Best herding-run-admitting contract in case V	0.3790	1.0000	1.0000	7.1529

7.4 An Example of an Economy with Two Depositors

In this section, I present the model in a two-depositor, two-stage version, which is the simplest setting that allows for herding runs. Depositors' endowment, preference and storage technology, bank's technology and the demand deposit contract, and the arrival of the information about the asset returns and consumption are the same as in the main text. Depositor 1 receives signals about asset returns and about his consumption type at stage 1. He does not have the chance to revise his decision after observing the action of the other. But he can delay his decision until stage 2. Depositor 2 receives information about his consumption type at stage 2, but no information about productivity. If both depositors are active at stage 2, they will make decisions simultaneously. For convenience, the signal about asset returns is assumed to be perfect ($q = 1$). Because there are only two depositors, there is no need for depositor 2 to make a decision before he receives signals. Both depositors have equal probability to be depositor 1 *ex ante*.

The bank offers the demand deposit contract (c_1, λ), where λ is the amount in storage. If $c_1 > 1$, the depositor who withdraws second will not receive the full amount of c_1. So let $c_1(1)$ and $c_2(2)$ denote the payment received by depositors who withdraw first and second in period 1, respectively. Let $c_2(x_1 + x_2, R)$ denote the payment in period 2 conditional on the total withdrawals in period 1 and the realization of asset returns.

To comply with the assumption in section 3, given $c_1 < 1$, I let depositor 1 delay his decision until stage 2 and that depositor 2 cannot obtain any information from depositor

1's action at stage 1. Depositors play a simultaneous-move game if both are active at stage 2. I first illustrate the equilibrium given $c_1 \geq 1$, then the one given $c_1 < 1$.

7.4.1 Equilibrium given $c_1 \geq 1$

In this simplest setup, there is a unique perfect Bayesian equilibrium in the post-deposit game, given any contract that provides $c_1 \geq 1$. That is,

1. If $c_1 = 1$ and $\lambda = 2$, depositors 1 and 2 withdraw if and only if they are impatient. Depositor 1's belief is updated by the signal received. Depositor 2's belief does not change. This contract results in the same welfare level as in autarky.

2. If $c_1 > 1$ or $\lambda \neq 2$, depositor 1 withdraws if he is impatient and/or a low signal is received and does not withdraw otherwise. Depositor 2 has the updated belief $P_{\widetilde{L}}(p_0)$ $(P_H(p_0) = 1)$ if depositor 1 withdraws (does not withdraw). Depositor 2 withdraws if he is impatient and/or his posterior belief at stage 2 is below the cutoff belief $\widehat{p}_2(x_1)$, where $\widehat{p}_2(x_1)$ solves

$$\widehat{p}_2(x_1)\, u\left(c_2\left(x_1, \overline{R}\right)\right) + \left(1 - \widehat{p}_2(x_1)\right) u\left(c_2\left(x_1, \underline{R}\right)\right) = u\left(c_1\left(x_1 + 1\right)\right). \tag{30}$$

   The left-hand side of (30) is the expected payoff for depositor 2 if he waits until period 2, and the right-hand side is the utility if he withdraws immediately. Note that $\widehat{p}_2$ is contingent on x_1, because depositor 2's expected payoff varies with depositor 1's action.

A herding run occurs in case 2 if depositor 1 withdraws at stage 1 and depositor 2 withdraws at stage 2 regardless of his consumption type.

Given $c_1 \geq 1$, an acceptable contract must satisfy the following condition: If the productivity is known to be high, both depositors are willing to wait *ex ante.* That is,

$$\begin{aligned} &\alpha^2\left(0.5u\left(c_1(1)\right) + 0.5u\left(c_1(2)\right)\right) + (1-\alpha)^2 u\left(c_2\left(0, \overline{R}\right)\right) + \\ &+2\alpha(1-\alpha)\left[0.5u\left(c_1(1)\right) + u\left(0.5c_2\left(1, \overline{R}\right)\right)\right] \geq u(1). \end{aligned} \tag{31}$$

Note that if a high signal is received, depositor 1 have the incentive to wait if he can convey the high signal to depositor 2 because

$$\alpha u\left(c_2\left(1,\overline{R}\right)\right)+(1-\alpha)\,u\left(c_2\left(0,\overline{R}\right)\right)\geq u\,(1)$$

by (31).

7.4.2 Equilibrium given $c_1 < 1$

When $c_1 < 1$ is provided ($c_1(1) = c_1(2) = c_1$), depositor 1 withdraws at stage 1 if he is impatient. If depositor 1 has withdrawn, depositor 2 withdraws at stage 2 if $p_0 u\left(c_2\left(1,\overline{R}\right)\right) + (1-p_0)\,u\left(c_2\left(1,\underline{R}\right)\right) < u\left(c_1\right)$ and/or he is impatient, and he does not otherwise. If depositor 1 is still active (i.e. he is a patient type), depositors 1 and 2 play a simultaneous-move game at stage 2, in which depositor 1 knows the portfolio status but does not know depositor 2's type, whereas depositor 2 does not know the portfolio status but knows depositor 1 is patient. There exist Bayesian Nash equilibria. There are four possible equilibrium outcomes, depending on the parameters and contract.

1. $\alpha u\left(c_2\left(1,\underline{R}\right)\right)+(1-\alpha)\,u\left(c_2\left(0,\underline{R}\right)\right) < u\left(c_1\right)$ and $p_0 u\left(c_2\left(0,\overline{R}\right)\right)+(1-p_0)\,u\left(c_2\left(1,\underline{R}\right)\right) \geq u\left(c_1\right)$: Depositor 1 withdraws if he has received a low signal, and does not otherwise. Depositor 2 withdraws if he is impatient and does not withdraw otherwise.

2. $u\left(c_2\left(1,\underline{R}\right)\right) < u\left(c_1\right)$ and $p_0 u\left(c_2\left(0,\overline{R}\right)\right) + (1-p_0)\,u\left(c_2\left(1,\underline{R}\right)\right) < u\left(c_1\right)$: Depositor 1 withdraws if he has received a low signal and does not withdraw otherwise. Depositor 2 withdraws.

3. $u\left(c_2\left(1,\underline{R}\right)\right) \geq u\left(c_1\right)$ and $p_0 u\left(c_2\left(0,\overline{R}\right)\right) + (1-p_0)\,u\left(c_2\left(0,\underline{R}\right)\right) < u\left(c_1\right)$: Depositor 1 does not withdraw. Depositor 2 withdraws.

4. $\alpha u\left(c_2\left(1,\underline{R}\right)\right)+(1-\alpha)\,u\left(c_2\left(0,\underline{R}\right)\right) \geq u\left(c_1\right)$ and $p_0 u\left(c_2\left(0,\overline{R}\right)\right)+(1-p_0)\,u\left(c_2\left(0,\underline{R}\right)\right) \geq u\left(c_1\right)$: Depositor 1 does not withdraw. Depositor 2 withdraws if he is impatient and does not withdraw otherwise.

Note that there exists multiple equilibria given some parameter values. Also note that depositor 1 always has incentive to wait if he has received a high signal as $c_1 < 1$ and $c_2\left(1, \overline{R}\right) > 1$.

7.4.3 A Numeric Example

The following utility function and parameters are used in the example: $u\left(c\right) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$; $\overline{R} = 1.25$, $\underline{R} = 0.95$, $p_0 = 0.95$; $q = 1$; $\alpha = 0.05$.

Table 3: Optimal contract - two-depositor, two-stage

	c_1	λ	$w_0\left(p_0\right)$
Best contract that provides $c_1 > 1$ or $\lambda \neq 2$	1	0	7.3439*
Contract that provides $c_1 = 1$ and $\lambda = 2$ (Autarky)	1	2	7.1529
Best contract that provides $c_1 < 1$	1.0000	0	7.3439

The optimal contract in this example provides $c_1 = 1$ and $\lambda = 0$. Because the liquidity demand is small (α is small) and the asset returns are likely to be high, the bank invests all resources in the productive assets. Given the optimal contract, depositor 1 withdraws at stage 1 if and only if a low signal is received or he is impatient, depositor 2 withdraws at stage 2 if depositor 1 has withdrawn at stage 1 or he is impatient, and does not otherwise. When asset returns are low, depositor 1's withdrawal forces the bank to liquidate all its assets. Depositor 2 benefits from early liquidation because it mitigates future losses, although he himself has no private information about asset returns. The best contract in the category of $c_1 < 1$ provides c_1 very close to 1, and the bank invests all resources in the asset. Given such a contract, there exists a unique equilibrium in which depositor 1 withdraws at stage 1 if and only if he is impatient, and he withdraws at stage 2 if and only if he has received a low signal; depositor 2 does not withdraw at stage 1, and he withdraws at stage 2 if and only if he is impatient.