

Suboptimal Choice and Aggregation

Jun Liu

UCSD

Individual Choice and Test of Asset Pricing Model

- ▶ In theory, the marginal utility of every agent in the economy should be the pricing kernel. We can test asset pricing models by using consumption data of individual agents.
- ▶ In reality, asset pricing models are tested using aggregate consumption data.
- ▶ One reason: individual choice may be suboptimal choice.
- ▶ Hopefully, the suboptimality is random and cancel out in aggregate.

What I do

1. Propose a plausible model of suboptimal choice.
2. Solve the randomness of choice in closed form.
3. Show that the aggregate consumption is not the optimal choice thus the randomness of suboptimality does not “average out.”

The Standard Utility Theory

1. Choice space C
2. Utility $U(C)$
3. Optimal choice maximizes utility $C^* = \operatorname{argmax}_C U(C)$.

A Model of Suboptimal Choice

- ▶ Any feasible C can be chosen.
- ▶ The probability density function $f(C)$ for C to be chosen is proportional $\exp\left(\frac{U(C)}{h}\right)$.
- ▶ The higher the utility of a choice C , the higher the probability for C to be chosen. The optimal consumption has a highest probability to be chosen.

Steepest Decent and Optimal Choice

- ▶ When $h \rightarrow 0$, $f(C) \propto \delta(C - C^*)$.

This is due to the mathematical theorem of Steepest Decent.

- ▶ When $h \rightarrow \infty$, $f(C)$ is a constant.

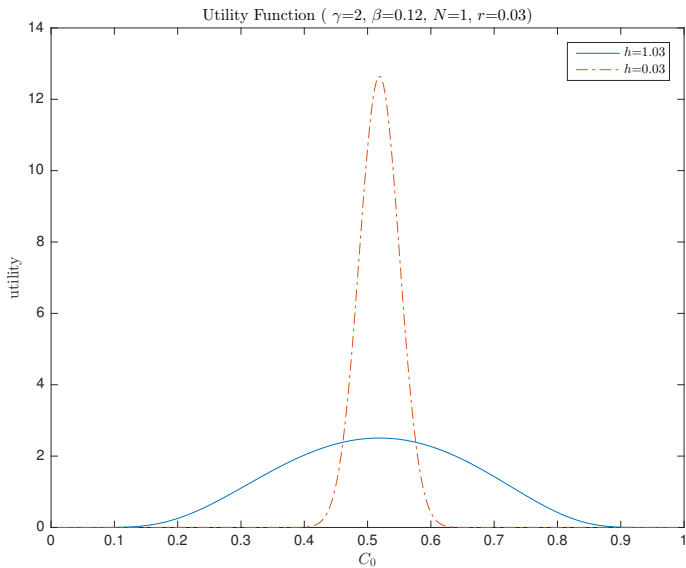


Figure 1: The probability density function.

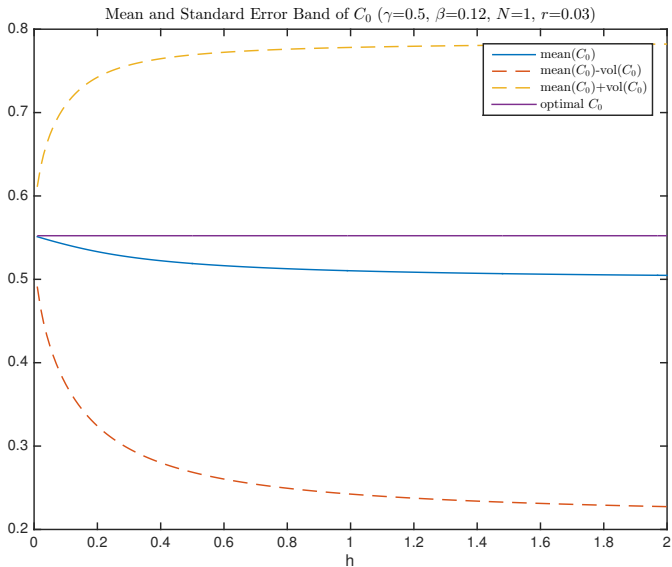


Figure 2: The optimal C_0 is higher than the mean of C_0 and is within the standard error band.

Log Utility: Probability Density Function

- ▶ We can get closed-form solution in the case of Log-utility,

$$U = \ln C_0 + e^{-\beta} \ln C_1.$$

- ▶ The probability is proportional to ($\alpha = e^{-\beta}$)

$$e^{\frac{1}{h}(\ln C_0 + \alpha \ln C_1)} = C_0^{\frac{1}{h}} C_1^{\frac{\alpha}{h}} = e^{\frac{r\alpha}{h}} C_0^{\frac{1}{h}} (W_0 - C_0)^{\frac{\alpha}{h}}$$

which can be written as

$$e^{\frac{r\alpha}{h}} W_0^{\frac{1+\alpha r}{h}} \hat{C}_0^{\frac{1}{h}} (1 - \hat{C}_0)^{\frac{\alpha}{h}}$$

with $\hat{C}_0 \equiv C_0/W_0$.

- ▶ The probability density is given by

$$f(\hat{C}_0) = \frac{\hat{C}_0^{\frac{1}{h}} (1 - \hat{C}_0)^{\frac{\alpha}{h}}}{\int_0^1 \hat{C}_0^{\frac{1}{h}} (1 - \hat{C}_0)^{\frac{\alpha}{h}} d\hat{C}_0} = \frac{\hat{C}_0^{\frac{1}{h}} (1 - \hat{C}_0)^{\frac{\alpha}{h}}}{B(\frac{1}{h} + 1, \frac{\alpha}{h} + 1)}$$

where $B(x, y)$ is the beta function.

Beta Function

- ▶ Definition

$$B(x, y) = \int_0^1 C^{x-1}(1 - C)^{y-1} dC.$$

- ▶ Important Property

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

with

$$\Gamma(x + 1) = x\Gamma(x).$$

Log Utility: Mean

- ▶ The mean of \hat{C}_0 is given by

$$\int_0^1 f(\hat{C}_0) \hat{C}_0 d\hat{C}_0 = \frac{B(\frac{1}{h} + 2, \frac{\alpha}{h} + 1)}{B(\frac{1}{h} + 1, \frac{\alpha}{h} + 1)}$$

Using the recursive relation of beta function, we get

$$E[\hat{C}_0] = \frac{\frac{1}{h} + 1}{\frac{1}{h} + 1 + \frac{\alpha}{h} + 1} = \frac{1 + h}{1 + \alpha + 2h}.$$

- ▶ The optimal C_0^* is

$$C_0^* = \frac{1}{1 + \alpha}.$$

As long as $h > 0$,

$$E[\hat{C}_0] < C_0^*.$$

Thus

$$E[\hat{C}_1] > C_1^*.$$

Log Utility: Variance

- ▶ The second moment of \hat{C}_0 is given by

$$E[\hat{C}_0^2] = \int_0^1 f(\hat{C}_0) \hat{C}_0^2 d\hat{C}_0 = \frac{B(\frac{1}{h} + 3, \frac{\alpha}{h} + 1)}{B(\frac{1}{h} + 1, \frac{\alpha}{h} + 1)}$$

Using the recursive relation of beta function, we get

$$E[\hat{C}_0^2] = \frac{(\frac{1}{h} + 2)(\frac{1}{h} + 1)}{(\frac{1}{h} + 2 + \frac{\alpha}{h} + 1)(\frac{1}{h} + 1 + \frac{\alpha}{h} + 1)}.$$

- ▶ The variance of \hat{C}_0 is

$$\left(\frac{(\frac{1}{h} + 2)}{(\frac{1}{h} + 2 + \frac{\alpha}{h} + 1)} - \frac{(\frac{1}{h} + 1)}{(\frac{1}{h} + 1 + \frac{\alpha}{h} + 1)} \right) \frac{(\frac{1}{h} + 1)}{(\frac{1}{h} + 1 + \frac{\alpha}{h} + 1)}$$

which equals to

$$\frac{(\frac{\alpha}{h} + 1)(\frac{1}{h} + 1)}{(\frac{1}{h} + 2 + \frac{\alpha}{h} + 1)(\frac{1}{h} + 1 + \frac{\alpha}{h} + 1)^2}.$$

- ▶ The variance goes to 0 as $h \rightarrow 0$ and $1/2$ as $h \rightarrow \infty$.
- ▶ As long as $h > 0$, the variance of C_0 is greater than 0.

Aggregation: Assumptions

1. Economy has N agents.
2. All agents have the same γ and β and h and initial wealth W_0 (all agents are identical “ex ante.”)
3. Different agents represent different independent draws from the distribution.

Equilibrium

- ▶ Market clearing

$$\frac{1}{N} \sum_{n=1}^N C_{tn} = D_t$$

where D_t is the per-capita dividend at time $t = 0, 1$.

- ▶ From our assumption,

$$\bar{C}_t = E[C_t] = W_0 E[\hat{C}_t],$$

as long as N is large enough.

- ▶ The above equations yield

$$(D_0 + e^{-r} D_1) \frac{1 + h}{1 + \alpha + 2h} = D_0.$$

- ▶ The equilibrium interest rate is given by

$$e^r = \frac{D_1}{D_0} \frac{1 + h}{e^{-\beta} + h}.$$

Properties of Equilibrium Interest Rate

- ▶ $h = 0$, the standard result.
- ▶ The equilibrium interest rate is lower than the interest rate in the standard model.
- ▶ $h = \infty$, $r = \ln(D_1/D_0)$.

Risky Case with Two States

- ▶ There are two dates, $t = 0$ and $t = 1$.
- ▶ There will be two states at $t = 1$, u and d .
- ▶ In this case, the budget constraint becomes

$$C_0 + p_u C_u + p_d C_d = W_0,$$

where p_u and p_d are state prices for u and d state respectively.

- ▶ The expected utility is

$$U = \ln C_0 + e^{-\beta}(\pi_u \ln C_u + \pi_d \ln C_d),$$

where π_u and π_d are probability of u and d state respectively..

Probability Density Function

- ▶ The probability is proportional to

$$e^{U/h} = C_0^{\frac{1}{h}} C_u^{\frac{\alpha\pi u}{h}} C_d^{\frac{\alpha\pi d}{h}} = (W_0 - p_u C_u - p_d C_d)^{\frac{1}{h}} C_u^{\frac{\alpha\pi u}{h}} C_d^{\frac{\alpha\pi d}{h}}$$

which can be written as

$$W_0^{\frac{1+\alpha\pi u+\alpha\pi d}{h}} p_u^{\frac{\alpha\pi u}{h}} p_d^{\frac{\alpha\pi d}{h}} (1 - \hat{C}_u - \hat{C}_d)^{\frac{1}{h}} \hat{C}_u^{\frac{\alpha\pi u}{h}} \hat{C}_d^{\frac{\alpha\pi d}{h}}$$

where $\hat{C}_u \equiv \frac{p_u C_u}{W_0}$ and $\hat{C}_d \equiv \frac{p_d C_d}{W_0}$.

- ▶ The probability density is

$$\frac{(1 - \hat{C}_u - \hat{C}_d)^{\frac{1}{h}} \hat{C}_u^{\frac{\alpha\pi u}{h}} \hat{C}_d^{\frac{\alpha\pi d}{h}}}{\int_{\{\hat{C}_u + \hat{C}_d \leq 1\}} (1 - \hat{C}_u - \hat{C}_d)^{\frac{1}{h}} \hat{C}_u^{\frac{\alpha\pi u}{h}} \hat{C}_d^{\frac{\alpha\pi d}{h}} d\hat{C}_u d\hat{C}_d}.$$

Note that

$$\int_{\{\hat{C}_u + \hat{C}_d \leq 1\}} (1 - \hat{C}_u - \hat{C}_d)^{\frac{1}{h}} \hat{C}_u^{\frac{\alpha\pi u}{h}} \hat{C}_d^{\frac{\alpha\pi d}{h}} d\hat{C}_u d\hat{C}_d = B(1 + \frac{1}{h}, 1 + \frac{\alpha\pi u}{h}, 1 + \frac{\alpha\pi d}{h}).$$

where $B(x_1, x_2, x_3)$ is the Dirichlet function.

Equilibrium

- ▶ The market clearing condition is

$$E[C_u] = D_u$$

and

$$E[C_d] = D_d$$

which can be written as

$$E[\hat{C}_u] = \hat{D}_u$$

and

$$E[\hat{C}_d] = \hat{D}_d$$

where $\hat{D}_u = \frac{p_u D_u}{W_0}$ and $\hat{D}_d = \frac{p_d D_d}{W_0}$.

Market Price

The expectation

$$E[\hat{C}_u] = \frac{1 + \frac{\pi_u \alpha}{h}}{3 + \frac{1 + \alpha(\pi_u + \pi_d)}{h}} = \frac{h + \alpha \pi_u}{3h + 1 + \alpha}.$$

So we have

$$\frac{h + \alpha \pi_u}{3h + 1 + \alpha} = \frac{p_u D_u}{W_0}.$$

Note that $W_0 = D_0 + p_u D_u + p_d D_d$, we get

$$(h + \alpha \pi_u)(D_0 + p_u D_u + p_d D_d) = (3h + 1 + \alpha)p_u D_u.$$

Similarly,

$$(h + \alpha \pi_d)(D_0 + p_u D_u + p_d D_d) = (3h + 1 + \alpha)p_d D_d.$$

Adding the two equations, we get

$$(2h + \alpha)(D_0 + p_u D_u + p_d D_d) = (3h + 1 + \alpha)(p_u D_u + p_d D_d).$$

We have

$$p_u D_u + p_d D_d = \frac{(2h + \alpha)D_0}{1 + h}.$$

The State Prices

- ▶ The state prices are given by

$$p_u = \frac{h + \alpha\pi_u}{1 + h} \frac{D_0}{D_u}$$

and

$$p_d = \frac{h + \alpha\pi_d}{1 + h} \frac{D_0}{D_d}.$$

- ▶ In the standard model,

$$p_u = \alpha\pi_u \frac{D_0}{D_u}$$

and

$$p_d = \alpha\pi_d \frac{D_0}{D_d}.$$

The Pricing Kernel

- ▶ The pricing kernel is given by

$$M_u = \frac{h + \alpha\pi_u}{(1 + h)\pi_u} \frac{D_0}{D_u}$$

and

$$M_d = \frac{h + \alpha\pi_d}{(1 + h)\pi_d} \frac{D_0}{D_d}.$$

- ▶ In the standard model,

$$p_u = \alpha \frac{D_0}{D_u}$$

and

$$p_d = \alpha \frac{D_0}{D_u}.$$

The Interest Rate and Equity Premium

- ▶ The riskless rate is

$$R_f^{-1} = \frac{h + \alpha\pi_u}{(1 + h)D_u} D_0 + \frac{h + \alpha\pi_d}{(1 + h)D_d} D_0.$$

- ▶ The price of the aggregate market is

$$S = p_u D_u + p_d D_d = \frac{2h + \alpha}{1 + h} D_0.$$

- ▶ The expected return of the market is

$$\mu = \frac{\pi_u D_u + \pi_d D_d}{\frac{(2h + \alpha)D_0}{1 + h}}.$$

- ▶ The risk premium is

$$\frac{1 + h}{D_0} \left(\frac{\pi_u D_u + \pi_d D_d}{2h + \alpha} - \left(\frac{h + \alpha\pi_u}{D_u} + \frac{h + \alpha\pi_d}{D_d} \right)^{-1} \right).$$

Equity Risk Premium

- ▶ The risk premium is

$$\frac{1+h}{D_0} \left(\frac{\pi_u D_u + \pi_d D_d}{2h + \alpha} - \left(\frac{h + \alpha \pi_u}{D_u} + \frac{h + \alpha \pi_d}{D_d} \right)^{-1} \right).$$

- ▶ This can be re-written as

$$\frac{1}{R_f} \frac{\pi_u \pi_d (D_u - D_d)^2}{D_u D_d} \frac{\alpha}{2h + \alpha} \left(1 + \frac{(\pi_u D_u - \pi_d D_d)h}{\alpha \pi_u \pi_d (D_u - D_d)} \right).$$

where

$$\frac{1}{R_f} \frac{\pi_u \pi_d (D_u - D_d)^2}{D_u D_d}$$

is the equity premium in the standard CCAPM.

- ▶ Assuming that $2\pi_d < 1$ and $D_d < D_u$, the factor

$$\frac{\alpha}{2h + \alpha} \left(1 + \frac{(\pi_u D_u - \pi_d D_d)h}{\alpha \pi_u \pi_d (D_u - D_d)} \right)$$

is greater than 1, thus the equity premium puzzle is exacerbated.

Extensions

- ▶ General γ .
- ▶ More than 1 period.
- ▶ More than 2 risky states.

Conclusions

- ▶ A plausible model of suboptimal choice.
- ▶ The choice is random.
- ▶ The randomness in suboptimality does not “average out.”